Sequences Solutions

1) (a) (i)$U_2 - U_1 = \frac{1}{8} - 1 = -\frac{7}{8}$

$U_3 - U_2 = -\frac{3}{4} - \frac{1}{8} = -\frac{7}{8}$

$U_4 - U_3 = -\frac{13}{8} - \left(-\frac{3}{4}\right) \quad = -\frac{7}{8}$

$\Rightarrow$ A common difference exists

$\Rightarrow$ (i) is an A.P.

(ii)As the sign of the terms oscillate (b) cannot be an A.P.

$\frac{U_2}{U_1} = \frac{-\frac{1}{8}}{1} = -\frac{1}{8}$

$\frac{U_3}{U_2} = \frac{\frac{3}{4}}{-\frac{1}{8}} = -6$

$\Rightarrow$ (ii) is not a G.P. either.

(iii)$\frac{U_2}{U_1} = \frac{-\frac{1}{8}}{1} = -\frac{1}{8}$

$\frac{U_3}{U_2} = \frac{\frac{1}{64}}{-\frac{1}{8}} = -\frac{1}{8}$

$\frac{U_4}{U_3} = \frac{-\frac{1}{512}}{\frac{1}{64}} = -\frac{1}{8}$

$\Rightarrow$ A common ratio exists

$\Rightarrow$ (iii) is a G.P.

(b) For the A.P. (i)

$a = 1$ $\quad d = -\frac{7}{8}$

Sum n terms of an A.P. is

$$S_n = \frac{n}{2}\left[2a + (n-1)d\right]$$

Sum of 6 terms $\quad S_6 = \frac{6}{2}\left[2\times 1 + (6-1)\left(\frac{-7}{8}\right)\right]$

$$S_6 = -7\frac{1}{8}$$

(c) For the G.P. (iii)

$a = 1 \qquad r = -\frac{1}{8}$

$|r| < 1$

$\therefore S_\infty = \frac{a}{1-r} = \frac{1}{1-\left(-\frac{1}{8}\right)} = \frac{1}{1\frac{1}{8}} = \frac{1}{\frac{9}{8}} \qquad S_\infty = \frac{8}{9}$

2) The sequence may be written $1, \left(\frac{1}{3}\right), \left(\frac{1}{3}\right)^2, \left(\frac{1}{3}\right)^3, \ldots\ldots$

$\therefore$ The r^{th} term $U_r = \frac{1}{3^{r-1}}$

As r increases indefinitely the terms get closer to 0

$\Rightarrow$ The sequence converges to 0

3) $U_1 = (-1)^1 + \frac{1}{1+1} = -\frac{1}{2} \qquad U_2 = (-1)^2 + \frac{1}{2+1} = 1\frac{1}{3}$

$U_3 = (-1)^3 + \frac{1}{3+1} = -\frac{3}{4} \qquad U_4 = (-1)^4 + \frac{1}{4+1} = 1\frac{1}{5}$

The sequence does not tend to a fixed number so is neither convergent nor divergent.
It is oscillatory.

4) $U_1 = 7 - \frac{1}{1} = 6 \qquad U_2 = 7 - \frac{1}{2} = 6\frac{1}{2} \qquad U_3 = 7 - \frac{1}{3} = 6\frac{2}{3} \qquad U_4 = 7 - \frac{1}{4} = 6\frac{3}{4}$

As n increases indefinitely the sequence is convergent to 7.

5) (a) $a_1 = 4(1) + 2 = 6$
$a_2 = 4(2) + 2 = 10$
$a_3 = 4(3) + 2 = 14$
$a_4 = 4(4) + 2 = 18$

The sequence 6, 10, 14, 18, …. is an arithmetic progression, which is divergent

(b) Sum of n terms of an arithmetic series is

$S_n = \frac{n}{2}\left[2a + (n-1)d\right] \qquad a = 6 \quad d = 4$

$$\Rightarrow \sum_{r=1}^{12}(4r+2) = \frac{12}{2}\left[2\times 6 + (12-1)\times 4\right]$$

$$\sum_{r=1}^{12}(4r+2) = 336$$

(c) If $\sum_{r=1}^{n}(4r+2) = 966$

$$\frac{n}{2}\left[2\times 6+(n-1)\times 4\right] = 966$$

$$\frac{n}{2}(4n+8) = 966$$

$$\frac{{}^{2}\cancel{4}n}{\cancel{2}}(n+2) = 966$$

$\div 2$ $\quad n^2 + 2n = 483$

$$n^2 + 2n - 483 = 0$$

$$(n+23)(n-21) = 0$$

$$n + 23 = 0 \quad , \quad n - 21 = 0$$

$$n = -23 \qquad n = 21$$

As n is a positive integer $n = 21$

6) $U_1 = 2$ $\quad U_2 = 3 - 2^2 = -1$ $\quad U_3 = 3 - (-1)^2 = 2$ $\quad U_4 = 3 - 2^2 = -1$ $\quad U_5 = 3 - (-1)^2 = 2$

(a) The sequence 2, –1, 2, –1, 2,…. is not convergent or divergent. It is oscillatory and periodic with period of 2

(b) $U_1 = \dfrac{\sqrt{13}-1}{2}$

$$U_2 = 3 - \left(\frac{\sqrt{13}-1}{2}\right)^2 = 3 - \frac{(13-2\sqrt{13}+1)}{4} = 3 - \frac{(14-2\sqrt{13})}{4} = 3 - \frac{\cancel{2}(7-\sqrt{13})}{\cancel{4}2} = 3 - \frac{(7-\sqrt{13})}{2}$$

$U_2 = \dfrac{\sqrt{13}-1}{2}$ $\quad \Rightarrow U_1 = U_2 \quad \Rightarrow$ All terms of the sequence will be the same

7) $a_1 = 12$ $\quad a_2 = 6 + \frac{1}{4}(12) = 9$ $\quad a_3 = 6 + \frac{1}{4}(9) = 8\frac{1}{4}$ $\quad a_4 = 6 + \frac{1}{4}\left(\frac{33}{4}\right) = 8\frac{1}{16}$ $\quad a_5 = 6 + \frac{1}{4}\left(\frac{129}{16}\right) = 8\frac{1}{64}$

(a) The sequence 12, 9, $8\frac{1}{4}$, $8\frac{1}{16}$, $8\frac{1}{64}$,....converges to 8. It is not oscillatory or periodic

(b) $a_2 = \left(6 + \frac{1}{4}a_1\right)$ $\qquad a_3 = 6 + \frac{1}{4}\left(6 + \frac{1}{4}a_1\right) = \frac{15}{2} + \frac{1}{16}a_1$

$$a_4 = 6 + \left(\frac{15}{2} + \frac{1}{16}a_1\right)\frac{1}{4} = 6 + \frac{15}{8} + \frac{1}{64}a_1 = \frac{1}{64}a_1 + \frac{63}{8} \quad \Rightarrow \quad k = \frac{63}{8} = 7\frac{7}{8}$$

8) $a_1 = 2$ $a_2 = 2 + 4 = 6$ $a_3 = 6 + 4 = 10$ $a_4 = 10 + 4 = 14$ $a_5 = 14 + 4 = 18$

(a) The sequence 2, 6, 10, 14, 18, …. is an arithmetic sequence with common difference 4.

(b) The Sum of n terms of an arithmetic series is

$$S_n = \frac{n}{2}\left[2a+(n-1)d\right]$$

$$\sum_{r=1}^{25}(a_r+4) = \frac{25}{2}\left(2\times 2+(25-1)\times 4\right)$$

$$\sum_{r=1}^{25}(a_r+4) = 1{,}250$$

9) $u_1 = 4$ $u_2 = -\frac{1}{2}\times 4 = -2$ $u_3 = -\frac{1}{2}(-2) = 1$ $u_4 = -\frac{1}{2}(1) = --\frac{1}{2}$ $u_5 = -\frac{1}{2}\left(-\frac{1}{2}\right) = \frac{1}{4}$

(a) The sequence $4,\ -2,\ 1,\ -\frac{1}{2},\ \frac{1}{4},\ldots$ is a geometric sequence with common ratio $-\frac{1}{2}$

It converges to 0 and is oscillatory.

(b) The sum of n terms of a geometric series is $S_n = \frac{a(1-r^n)}{(1-r)}$

$$\therefore \qquad \sum_{r=1}^{10} u_r = \frac{4\left(1-\left(-\frac{1}{2}\right)^{10}\right)}{1-\left(-\frac{1}{2}\right)} = 2\frac{85}{128}$$

10) (a) $t_1 = 2\times 1 + 5 = 7$ $t_2 = 2\times 2 + 5 = 9$ $t_3 = 2\times 3 + 5 = 11$ $t_4 = 2\times 4 + 5 = 13$

The sequence 7, 9, 11. 13, ….is an A.P.. It diverges.

(b) $U_1 = (-1)^1 + \frac{1}{1+2} = -\frac{2}{3}$ $U_2 = (-1)^2 + \frac{1}{2+2} = 1\frac{1}{4}$

$U_3 = (-1)^3 + \frac{1}{3+2} = -\frac{4}{5}$ $U_4 = (-1)^4 + \frac{1}{4+2} = 1\frac{1}{6}$

The sequence $-\frac{2}{3},\ 1\frac{1}{4},\ -\frac{4}{5},\ 1\frac{1}{6},\ldots$ is oscillating, not converging or diverging

(c) $U_1 = 6-\frac{4}{1} = 2$ $U_2 = 6-\frac{4}{2} = 4$ $U_3 = 6-\frac{4}{3} = 4\frac{2}{3}$ $U_4 = 6-\frac{4}{4} = 5$

The sequence $2,\ 4,\ 4\frac{2}{3},\ 5,\ldots$ converges to 6

(d) $U_1 = 0$ $U_2 = 0^2 - 1 = -1$ $U_3 = (-1)^2 - 1 = 0$ $U_4 = 0^2 - 1 = -1$
The sequence 0, −1, 0, −1, …oscillates and is periodic with period = 2