Quadratic Functions Solutions

1)

(a) By completing the square

$$x^2 + 4x + 7 \equiv (x + 2)^2 + a = x^2 + 4x + 4 + a$$

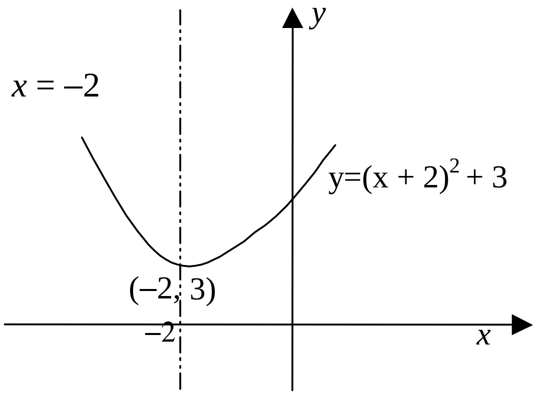
matching up no.s $\quad 7 \equiv 4 + a$

$\Rightarrow \quad a = 3$

(b) $\quad y = x^2 + 4x + 7$

$\Rightarrow \quad y = (x + 2)^2 + 3$

$\Rightarrow$ The function has a minimum of 3 where $x = -2$



The dashed line is the axis of symmetry which has equation $x = -2$

2)

(a) $\quad f : x \rightarrow \; = x^2 + 8x - 3$

By completing the square

$$f : x \rightarrow \; x^2 + 8x - 3 = (x - p)^2 + q = x^2 - 2px + p^2 + q$$

matching up x\`s $\quad 8x = -2px \qquad p = -4$

matching up no.s $\quad -3 = p^2 + q = 16 + q \qquad q = -19$

$$x^2 + 8x - 3 = (x + 4)^2 - 19$$

$\Rightarrow \quad p = -4 \quad q = -19$

(b) When $x = -4$, the function will have a minimum value of -19

3)

(a) By completing the square

$$x^2 + 4x + 14 = (x + p)^2 + q = x^2 + 2px + p^2 + q$$

matching up x\`s $\quad 4x = 2px \qquad p = 2$

matching up no\`s $\quad 14 = p^2 + q = 4 + q \qquad q = 10$

$\Rightarrow \quad p = 2 \qquad q = 10$

(b) $\Rightarrow$ Minimum value of $x^2 + 4x + 14$ is 10 which occurs when $x = -2$

(c) $\quad x^2 + 4x + k = 0$

$a = 1 \qquad b = 4 \qquad c = k$

For equal roots

$$b^2 - 4ac = 0$$

$$4^2 - 4 \times 1 \times k = 0$$

$$16 = 4k$$

$\Rightarrow \quad k = 4$

4)

(a) $f:x \to = 12 - 8x - x^2 = q - (x + p)^2 = -x^2 - 2px - p^2 + q$

matching up x`s $\quad -8x = -2px \quad p = 4$

matching up no`s $\quad 12 = -p^2 + q = -16 + q \quad q = 28$

$$12 - 8x - x^2 = 28 - (x + 4)^2$$

$\Rightarrow \quad p = 4, \quad q = 28$

(b) $f:x \to$ has a maximum value of 28 which occurs when $x = -4$

(c) Minimum value of $\dfrac{1}{12-8x-x^2}$ occurs when

$12 - 8x - x^2$ has a maximum value.

$\Rightarrow$ Minimum value of $\dfrac{1}{12-8x-x^2} = \dfrac{1}{28}$

5)

(a) By completing the square

$$x^2 - 8x - 1 \equiv (x + a)^2 + b = x^2 + 2ax + a^2 + b$$

matching up x`s $\quad -8x = 2ax \quad a = -4$

matching up no`s $\quad -1 = a^2 + b = 16 + b \quad b = -17$

$$x^2 - 8x - 1 \equiv (x - 4)^2 - 17$$

$\Rightarrow \quad a = -4 \quad b = -17$

Minimum value of $x^2 - 8x - 1$ is -17 which occurs when $x = 4$

(b) Maximum value of $\dfrac{1}{x^2-8x-1}$ occurs when $x^2 - 8x - 1$ has a minimum value

$\Rightarrow$Maximum value of $\dfrac{1}{x^2-8x-1} = \dfrac{1}{-17}$

Maximum value of $\dfrac{1}{x^2-8x-1} = -\dfrac{1}{17}$

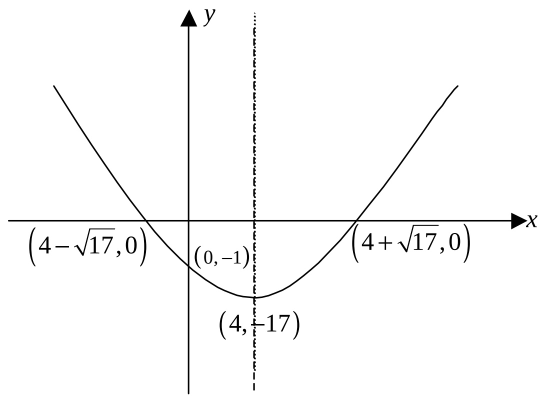
(c) To cut y axis $\quad x = 0$

$\Rightarrow$ Curve cuts y axis at $(0, -1)$

To cut x axis $\quad y = 0$

$$\begin{aligned} x^2 - 8x - 1 &= 0 \\ (x-4)^2 - 17 &= 0 \\ (x-4)^2 &= 17 \\ x - 4 &= \pm\sqrt{17} \\ x &= 4 \pm \sqrt{17} \end{aligned}$$

$\Rightarrow$ Curve cuts x axis at $\left(4-\sqrt{17}, 0\right), \left(4+\sqrt{17}, 0\right)$



The line of symmetry (dashed line) is $x = 4$

6)

Let v mph be Bill's speed.

$\Rightarrow$ Ashana's speed $= v + 2$

Time for Bill to travel 28 miles $= \dfrac{28}{v}$

Time for Ashana to travel 28 miles $= \dfrac{28}{(v+2)}$

Bill's time $-$ Ashana's time $=$ 20 minutes

$$\frac{28}{v} - \frac{28}{(v+2)} = \frac{1}{3}$$

$$\Rightarrow \quad \frac{28(v+2)-28v}{v(v+2)} = \frac{1}{3}$$

$$\Rightarrow \quad \frac{28(v+2-v)}{v(v+2)} = \frac{1}{3}$$

$$\Rightarrow \quad \frac{56}{v(v+2)} = \frac{1}{3}$$

Multiply both sides by $3v(v + 2)$

$$168 = v(v + 2)$$

$$\Rightarrow \quad v^2 + 2v - 168 = 0$$

$$(v + 14)(v - 12) = 0$$

$$v + 14 = 0 \quad \text{or} \quad v - 12 = 0$$

$$v = -14 \quad \text{or} \quad v = 12$$

$\Rightarrow$ Bill's speed $=$ 12mph

and Ashana's speed $=$ 14mph

7)

$$(k - 1)x^2 - 2(2k + 1)x + 4k + 9 = 0$$

$$a = (k - 1), \quad b = -2(2k + 1), \quad c = (4k + 9)$$

For equal roots $b^2 - 4ac = 0$

$$\Rightarrow [-2(2k + 1)]^2 - 4(k - 1)(4k + 9) = 0$$

$$4(2k + 1)^2 - 4(4k^2 + 5k - 9) = 0$$

$$\div 4 \quad (4k^2 + 4k + 1) - (4k^2 + 5k - 9) = 0$$

$$4k^2 + 4k + 1 - 4k^2 - 5k + 9 = 0$$

$-k + 10 = 0 \quad \Rightarrow \quad k = 10$

8)

(a) $f:x\rightarrow = 5 - 8x - x^2$

By completing the square

$f:x\rightarrow = 5 - 8x - x^2 = \alpha - (x + \beta)^2 = -x^2 - 2\beta x - \beta^2 + \alpha$

matching up x`s $\quad -8x = -2\beta x \quad\quad \beta = 4$

matching up no.`s $\quad 5 = -\beta^2 + \alpha = -16 + \alpha \quad\quad \alpha = 21$

$5 - 8x - x^2 = 21 - (x + 4)^2$

$\Rightarrow \quad \alpha = 21 \quad \beta = 4$

(b) $\Rightarrow$ Maximum value of $f:x\rightarrow$ is 21 which occurs when $x = -4$

(c) $y = 5 - 8x - x^2$

To cut x axis, $y = 0$

$\Rightarrow \quad 5 - 8x - x^2 = 0$

$\Rightarrow \quad 21 - (x + 4)^2 = 0$

$(x + 4)^2 = 21$

$x + 4 = \pm\sqrt{21}$

$x = -4 \pm \sqrt{21}$

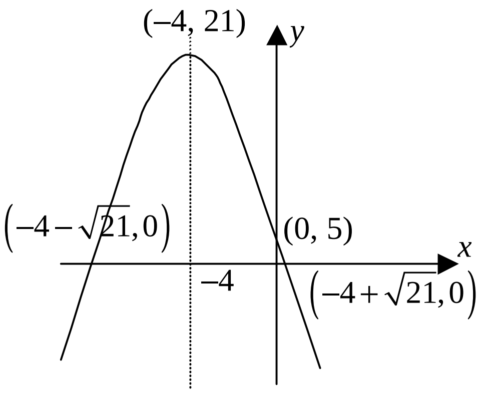
$\Rightarrow$Curve cuts the x axis at $\left(-4-\sqrt{21}, 0\right)$ and $\left(-4+\sqrt{21}, 0\right)$

To cut y axis, $x = 0$

$\Rightarrow \quad y = 5$

$\Rightarrow$ Curve cuts y axis at $(0, 5)$

The dashed line is the axis of symmetry which has equation $x = -4$



9)

(a) $(3a - 2)x^2 + 2ax + 1 = 0$

$a = (3a - 2) \quad b = 2a \quad c = 1$

For equal roots $\quad b^2 - 4ac = 0$

$(2a)^2 - 4(3a - 2) \times 1 = 0$

$4a^2 - 4(3a - 2) = 0$

$\div 4 \quad a^2 - 3a + 2 = 0$

$(a - 1)(a - 2) = 0$

$a - 1 = 0$ or $a - 2 = 0$

$a = 1$ or $a = 2$

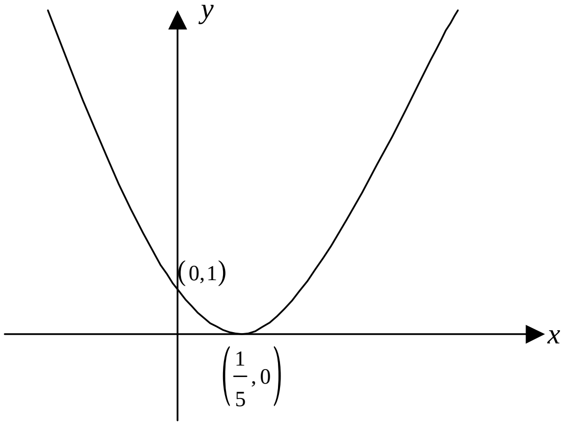
(b) $25x^2 - 10x + 1 = 0$

$a = 25 \quad b = -10 \quad c = 1$

$b^2 - 4ac = (-10)^2 - 4 \times 25 \times 1 = 100 - 100 = 0$

$\Rightarrow$ The equation has equal roots

$$25x^2 - 10x + 1 = 0$$
$$(5x-1)(5x-1) = 0$$
$$\Rightarrow \quad 5x - 1 = 0$$
$$x = \frac{1}{5}$$
$\Rightarrow$ Curve touches x axis at $\left(\frac{1}{5}, 0\right)$



To cut y axis $\quad x = 0$

$\Rightarrow \quad y = 1$

$\Rightarrow$ Curve cuts y axis at (0, 1)

10)

(a) By completing the square

$$5 + 8x - 2x^2 = a(x-b)^2 + c = ax^2 - 2abx + ab^2 + c$$

matching up x^2 s $\quad -2x^2 = ax^2 \qquad a = -2$

matching up x\`s $\quad 8x = -2abx$

$\qquad = 4bx \qquad b = 2$

matching up no\`s $\quad 5 = ab^2 + c$

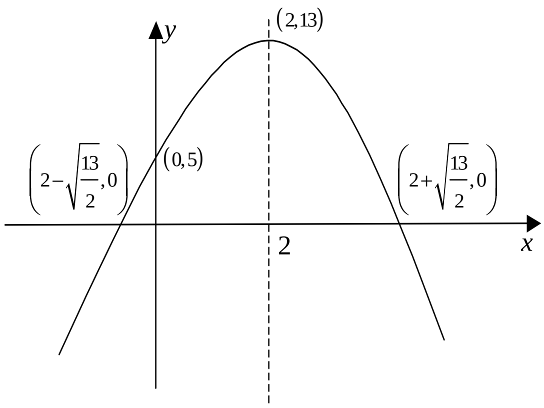
$\qquad = -8 + c \qquad c = 13$

So $\quad 5 + 8x - 2x^2 = 13 - 2(x-2)^2$

$\Rightarrow$ The maximum value of the function is 13 which occurs when $x = 2$

(b) To cut x axis $\quad y = 0$

$$5 + 8x - 2x^2 = 0$$
$$13 - 2(x-2)^2 = 0$$
$$(x-2)^2 = \frac{13}{2}$$
$$x - 2 = \pm\sqrt{\frac{13}{2}}$$
$$x = 2 \pm \sqrt{\frac{13}{2}}$$



$\Rightarrow$ Curve cuts x axis at $\left(2 - \sqrt{\frac{13}{2}}, 0\right)$ and $\left(2 + \sqrt{\frac{13}{2}}, 0\right)$

To cut y axis $\quad x = 0$

$\Rightarrow$ Curve cuts y axis at (0, 5)

Equation of axis of symmetry is $\quad x = 2$

(c) Minimum value of $\frac{1}{5 + 8x - 2x^2}$ occurs when $5 + 8x - 2x^2$ is a maximum.

$\Rightarrow \quad$ Minimum value $= \frac{1}{13}$