Equations of Tangents and Normals

1)

$$y = x^3 + 2x^2 - 4x + 6$$
$$\frac{dy}{dx} = 3x^2 + 4x - 4$$

(a) When $x = -3$

$$y = (-3)^3 + 2(-3)^2 - 4(-3) + 6$$
$$y = 9$$

(b) When $x = -3$

$$\frac{dy}{dx} = 3(-3)^2 + 4(-3) - 4$$
$$\frac{dy}{dx} = 11$$

(c) Equation of tangent at $P(-3, 9)$ is given by

$$y - y_1 = m(x - x_1)$$
$$y - 9 = 11(x - (-3))$$
$$y = 11x + 42$$

(d) ⇒The gradient of the tangent at $P(-3, 9) = 11$

⇒The gradient of the normal at $P(-3, 9) = -\frac{1}{11}$ using $m_1 m_2 = -1$

⇒Equation of normal at $P(-3, 9)$ is

$$y - 9 = -\frac{1}{11}\left(x - (-3)\right)$$

$\times 11$ $$11y - 99 = -x - 3$$

⇒ $$x + 11y = 96$$

2)

(a)

$$y = x^3 - x + 8$$
$$\frac{dy}{dx} = 3x^2 - 1$$

(b) When $x = -2$

$$y = (-2)^3 - (-2) + 8$$
$$y = 2$$

(c) When $x = -2$

$$\frac{dy}{dx} = 3(-2)^2 - 1$$
$$\frac{dy}{dx} = 11$$

(d) Equation of the tangent at $P\ (-2, 2)$ is given by

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y - 2 &= 11(x - (-2)) \\ y &= 11x + 24 \end{aligned}$$

(e) Find the other point on the curve where

$$\frac{dy}{dx} = 11$$

$$\Rightarrow \quad 3x^2 - 1 = 11$$

$$x^2 = 4$$

$$x = -2 \quad \text{or} \quad x = 2$$

$\Rightarrow$ The other point, Q, on the curve at which the tangent is parallel to

the tangent at $P\ (-2, 2)$ has

$$x \text{ co-ordinate} = 2$$

$$\Rightarrow \quad y = (2)^3 - 2 + 8 = 14$$

$\Rightarrow$ Q has co-ordinates $(2, 14)$

(f) Gradient of tangent at $Q\ (2, 14)$ = 11

$\Rightarrow$Gradient of normal at $Q\ (2, 14)$ =

$-\frac{1}{11}$ using $m_1 m_2 = -1$

$\Rightarrow$ Equation of the normal at $Q\ (2, 14)$ is

$$y - 14 = -\frac{1}{11}(x - 2)$$

$$\times 11 \qquad 11y - 154 = -x + 2$$

$$\Rightarrow \quad x + 11y = 156$$

3)

(a) To find the points of intersection, solve the following equations simultaneously

$$y = 6x - 5 \qquad \text{-\{1\}}$$

$$y = x^2 \qquad \text{-\{2\}}$$

$$\Rightarrow \quad x^2 = 6x - 5$$

$$\Rightarrow \quad x^2 - 6x + 5 = 0$$

$$(x - 1)(x - 5) = 0$$

$$\Rightarrow \quad x - 1 = 0 \quad \text{or} \quad x - 5 = 0$$

$$x = 1 \quad \text{or} \quad x = 5$$

$$y = 6(1) - 5 \qquad y = 6(5) - 5$$

$$y = 1 \qquad y = 25$$

$\Rightarrow$ The points of intersection are $P\ (1, 1)$ and $Q\ (5, 25)$

(b)

$$y = x^2$$
$$\frac{dy}{dx} = 2x$$

At the point P when $x = 1$ $\frac{dy}{dx} = 2(1) = 2$

Equation of the tangent at P (1, 1) is given by

$$y - y_1 = m(x - x_1)$$
$$y - 1 = 2(x - 1)$$
$$\Rightarrow \quad y = 2x - 1$$

At the point Q, when $x = 5$ $\frac{dy}{dx} = 2(5) = 10$

Equation of the tangent at Q (5, 25)

$$y - 25 = 10(x - 5)$$
$$y = 10x - 25$$

(c) For point of intersection of the tangents, solve the following equations simultaneously

$$y = 2x - 1 \quad -\{1\}$$
$$y = 10x - 25 \quad -\{2\}$$
$$\{2\} - \{1\} \quad 8x - 24 = 0$$
$$\Rightarrow \quad x = 3$$
$$\Rightarrow \quad y = 2(3) - 1$$
$$= 5$$

Co-ordinates of the point of intersection of the tangents are (3, 5)

4)

(a)

$$y = x^3 - 3x^2 - 12x - 4$$
$$\frac{dy}{dx} = 3x^2 - 6x - 12$$

(b) When the gradient of the curve $C = -3$

$$\frac{dy}{dx} = -3$$
$$\Rightarrow \quad 3x^2 - 6x - 12 = -3$$
$$\Rightarrow \quad 3x^2 - 6x - 9 = 0$$
$$\div 3 \quad x^2 - 2x - 3 = 0$$
$$\Rightarrow \quad (x + 1)(x - 3) = 0$$
$$\Rightarrow \quad x + 1 = 0 \text{ or } x - 3 = 0$$
$$x = -1 \text{ or } x = 3$$

When $x = -1$

$$y = (-1)^3 - 3(-1)^2 - 12(-1) - 4$$
$$= 4$$

When $x = 3$ $\quad y = (3)^3 - 3(3)^2 - 12(3) - 4$

$= -40$

$\Rightarrow$ The co-ordinates of the points on the curve C are $(-1,4)$ and $(3,-40)$

Equation of the tangent at the point $(-1, 4)$ is given by

$$y - y_1 = m(x - x_1)$$
$$y - 4 = -3(x - (-1))$$
$$y = -3x + 1$$

Equation of the tangent at the point $(3, -40)$ is

$$y - (-40) = -3(x - 3)$$
$$y = -3x - 31$$

5) (a) $\quad y = x^2 - 4x + 8$

$$\frac{dy}{dx} = 2x - 4$$

When $x = 1$, $\quad \frac{dy}{dx} = 2(1) - 4$

$= -2$

$\Rightarrow$ Gradient of the curve $= -2$ at the point where $x = 1$

(b) Gradient of normal $= \frac{-1}{-2}$ using $m_1 m_2 = -1$

$= \frac{1}{2}$

When $x = 1$, $\quad y = (1)^2 - 4(1) + 8$

$= 5$

Equation of the normal at the point $(1, 5)$ is given by

$$y - y_1 = m(x - x_1)$$
$$y - 5 = \frac{1}{2}(x - 1)$$

$\times 2$ $\quad 2y - 10 = x - 1$

$$2y = x + 9$$

(c) To find the other point where the normal intersects with the curve, solve the following equations simultaneously

$2y = x + 9$ $\quad$ -{1}

$y = x^2 - 4x + 8$ $\quad$ -{2}

From {1} substitute $y = \frac{(x+9)}{2}$ in {2}

$$\frac{(x+9)}{2} = x^2 - 4x + 8$$

$\times 2$ $\quad x + 9 = 2x^2 - 8x + 16$

$$\Rightarrow \quad 2x^2 - 9x + 7 = 0$$
$$\Rightarrow \quad (2x - 7)(x - 1) = 0$$
$$\Rightarrow \quad 2x - 7 = 0 \quad , \quad x - 1 = 0$$
$$x = \frac{7}{2} \quad , \quad x = 1$$
$$\Rightarrow \quad y = \frac{\frac{7}{2} + 9}{2} \qquad y = 5$$
$$= \frac{25}{4}$$

$\Rightarrow$Co-ordinates of the other point where the normal intersects with the curve are $\left(3\frac{1}{2}, 6\frac{1}{4}\right)$

6) (a) For points of intersection solve the equations simultaneously

$$y = 2x + 1 \qquad -\{1\}$$
$$y = 3x^2 + 7x - 1 \qquad -\{2\}$$
$$\Rightarrow \quad 3x^2 + 7x - 1 = 2x + 1$$
$$3x^2 + 5x - 2 = 0$$
$$\Rightarrow \quad (3x - 1)(x + 2) = 0$$
$$\Rightarrow \quad 3x - 1 = 0 \quad \text{or} \quad x + 2 = 0$$
$$x = \frac{1}{3} \quad \text{or} \quad x = -2$$

When $x = \frac{1}{3}$, $\quad y = 3\left(\frac{1}{3}\right)^2 + 7\left(\frac{1}{3}\right) - 1$
$$= 1\frac{2}{3}$$

Co-ordinates of B are $\left(\frac{1}{3}, 1\frac{2}{3}\right)$

When $x = -2$ $\quad y = 3(-2)^2 + 7(-2) - 1$
$$= -3$$

Co-ordinates of A are $(-2, -3)$

(b)
$$\frac{dy}{dx} = 6x + 7$$

When $x = -2$, $\quad \frac{dy}{dx} = 6(-2) + 7$
$$= -12 + 7$$
$$= -5$$

Equation of the tangent at $A(-2, -3)$ is given by

$$y - y_1 = m(x - x_1)$$
$$y - (-3) = -5(x - (-2))$$
$$y + 3 = -5x - 10 \qquad y = -5x - 13$$

(c) When $x = \frac{1}{3}$, $\frac{dy}{dx} = 6\left(\frac{1}{3}\right) + 7$

$= 9$

$\Rightarrow$ Gradient of normal $= -\frac{1}{9}$ using $m_1 m_2 = -1$

Equation of normal at the point $B\left(\frac{1}{3}, 1\frac{2}{3}\right)$

$$y - \frac{5}{3} = -\frac{1}{9}\left(x - \frac{1}{3}\right)$$

$\times 27$ $\quad 27y - 45 = -3x + 1$

$3x + 27y = 46$

7) (a) $\frac{dy}{dx} = 3x^2 - 1$

(b) When $x = -1$, $y = (-1)^3 - (-1) + 6$

$y = 6$

(c) When $x = -1$ $\frac{dy}{dx} = 3(-1)^2 - 1$

$\frac{dy}{dx} = 2$

(d) Equation of the tangent at $P(-1, 6)$ is given by

$y - y_1 = m(x - x_1)$

$y - 6 = 2(x - (-1))$

$\Rightarrow$ $y = 2x + 8$

(e) $\frac{dy}{dx} = 2 \Rightarrow$ $3x^2 - 1 = 2$

$3x^2 = 3$

$x = -1$ or $x = 1$

$\Rightarrow x$ co-ordinate of $Q = 1$

When $x = 1$ $y = (1)^3 - (1) + 6$

$= 6$

$\Rightarrow$ Co-ordinates of Q are $(1, 6)$

(f) Gradient of normal at $Q(1, 6)$ $= -\frac{1}{2}$ using $m_1 m_2 = -1$

Equation of normal at $Q(1, 6)$

$$y - 6 = -\frac{1}{2}(x - 1)$$

$\Rightarrow$ $2y = -x + 13$

8) (a) When $x = -1$

$$y = (-1)^3 - 3(-1)^2 - 4(-1) + 12$$
$$y = 12$$

(b)

$$\frac{dy}{dx} = 3x^2 - 6x - 4$$

When $x = -1$, $$\frac{dy}{dx} = 3(-1)^2 - 6(-1) - 4$$
$$= 5$$

$\Rightarrow$Gradient of the curve at P = 5

(c) Equation of the tangent to the curve at the P (–1, 12) is given by

$$y - y_1 = m(x - x_1)$$
$$y - 12 = 5(x - (-1))$$
$$y = 5x + 17$$

(d) Gradient of the normal to the curve at the point

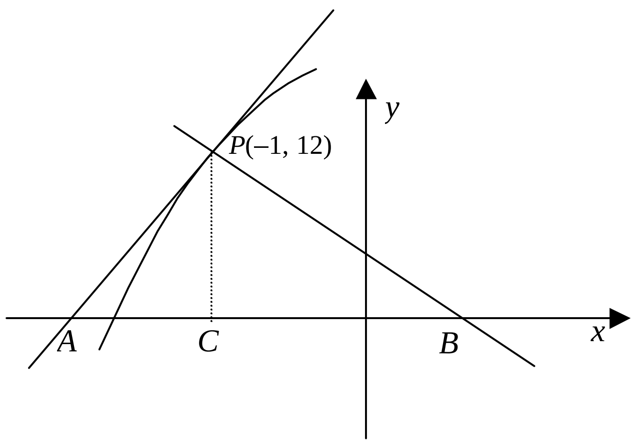
$$P(-1, 12) = -\frac{1}{5} \text{ using } m_1 m_2 = -1$$

Equation of the normal at the point $P(-1, 12)$

$$y - 12 = -\frac{1}{5}\left(x - (-1)\right)$$

$\times 5$ $$5y - 60 = -x - 1$$
$$5y = -x + 59$$

(e)



Tangent cuts x axis when $y = 0$

$\Rightarrow$ $$5x + 17 = 0$$
$$x = -3\frac{2}{5}$$

$\Rightarrow$ Co-ordinates of A are $\left(-3\frac{2}{5}, 0\right)$

Normal cuts x axis when $y = 0$

$$\Rightarrow \quad -x + 59 = 0$$

$$\Rightarrow \quad x = 59$$

$$AB = 59 - \left(-3\frac{2}{5}\right)$$

$$= 62\frac{2}{5}$$

$$\text{Height of triangle } ABP = PC$$

$$= 12$$

$$\text{Area of triangle } ABP = \frac{1}{2} \times AB \times PC$$

$$= \frac{1}{2} \times 62\frac{2}{5} \times 12$$

$$\text{Area of triangle } ABP = 374\frac{2}{5} \text{ sq units}$$

9) (a) Solve the equations simultaneously

$$y = 4$$

$$y = x^2 - 4x + 7$$

$$\Rightarrow \quad x^2 - 4x + 7 = 4$$

$$x^2 - 4x + 3 = 0$$

$$\Rightarrow \quad (x - 1)(x - 3) = 0$$

$$\Rightarrow \quad x - 1 = 0 \quad \text{or} \quad x - 3 = 0$$

$$x = 1 \quad \text{or} \quad x = 3$$

$\Rightarrow$ Co-ordinates of the points of intersection are (1, 4) and (3, 4)

(b)

$$\frac{dy}{dx} = 2x - 4$$

When $x = 1$

$$\frac{dy}{dx} = 2(1) - 4$$

$$= -2$$

$\Rightarrow$ Gradient of the normal at the point (1, 4) = $\frac{1}{2}$ using

$m_1 m_2 = -1$

Equation of the normal at the point (1, 4) is given by

$$y - y_1 = m(x - x_1)$$

$$y - 4 = \frac{1}{2}(x - 1)$$

$$\times 2 \quad 2y - 8 = x - 1$$

$$\Rightarrow \quad 2y = x + 7$$

When $x = 3$

$$\frac{dy}{dx} = 2(3) - 4$$

$$= 2$$

$\Rightarrow$ Gradient of the normal at the point $(3, 4) = -\frac{1}{2}$ using $m_1 m_2 = -1$

Equation of the normal at the point (3, 4) is

$$y - 4 = -\frac{1}{2}(x - 3)$$

$\times 2$ $\quad 2y - 8 = -x + 3$

$$2y = -x + 11$$

(c) For point of intersection of the normal, solve the equations simultaneously

$$2y = x + 7 \quad \text{-}\{1\}$$

$$2y = -x + 11 \quad \text{-}\{2\}$$

$\{1\}+\{2\}$ $\quad 4y = 18$

$$y = 4\frac{1}{2}$$

$\Rightarrow$ $\quad x = 2$

$\Rightarrow$ Point of intersection of normals is $\left(2, 4\frac{1}{2}\right)$

10) (a) $\quad \frac{dy}{dx} = 8 - 2x$

(b) If the tangent is parallel to the line $y = 2x + 5$

$\Rightarrow$ $\quad \frac{dy}{dx} = 2$

$\Rightarrow$ $\quad 8 - 2x = 2$

$\Rightarrow$ $\quad x = 3$

$\Rightarrow$ $\quad y = 8(3) - (3)^2 - 12$

$$= 3$$

Co–ordinates of P, the point of contact (3, 3)

Equation of the tangent at the point (3, 3) is given by

$$y - y_1 = m(x - x_1)$$

$$y - 3 = 2(x - 3)$$

$\Rightarrow$ $\quad y = 2x - 3$

(c) Gradient of normal at (3, 3) $= -\frac{1}{2}$ using $m_1 m_2 = -1$

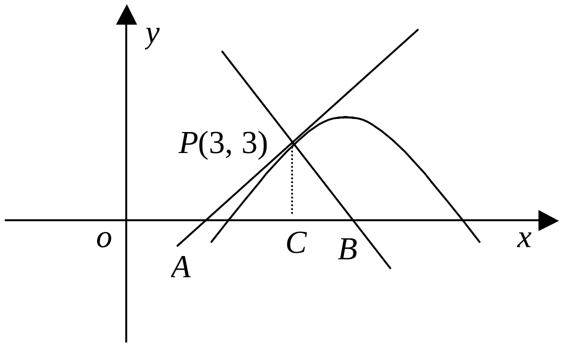
Equation of normal at $P(3, 3)$ is

$$y - 3 = -\frac{1}{2}(x - 3)$$

$\times 2$ $\quad 2y - 6 = -x + 3$

$$2y = -x + 9$$

(d)



For tangent to cut x axis $y = 0$

$$\Rightarrow \quad 2x - 3 = 0$$

$$\Rightarrow \quad x = 1\frac{1}{2}$$

Co-ordinates of $A\left(1\frac{1}{2}, 0\right)$

For normal to cut x axis, $y = 0$

$$\Rightarrow \quad -x + 9 = 0$$

$$x = 9$$

Co-ordinates of $B(9, 0)$

$$\Rightarrow \quad AB = 9 - 1\frac{1}{2} = 7\frac{1}{2}$$

Height of triangle, $PC = 3$

Area of triangle $APB = \frac{1}{2} \times AB \times PC$

$$= \frac{1}{2} \times \frac{15}{2} \times 3$$

Area of triangle $APB = 11\frac{1}{4}$ square units