Differentation of Powers of x Solutions

1)

$$\frac{dy}{dx} = 3x^2 + 12x + 15$$

Turning points occur when $\frac{dy}{dx} = 0$

$\Rightarrow \quad 3x^2 + 12x + 15 = 0$

$\div 3 \quad x^2 + 4x + 5 = 0$

$a = 1, \quad b = 4, \quad c = 5$

$b^2 - 4ac = 4^2 - 4 \times 1 \times 5$

$= -4 < 0$

$\Rightarrow$ The equation has no roots

$\Rightarrow$ The curve has no turning points

2) $\frac{dy}{dx} = 6x^2 + 6x - 36$

Gradient $= 0$

$\Rightarrow \quad 6x^2 + 6x - 36 = 0$

$\div 6 \quad x^2 + x - 6 = 0$

$(x + 3)(x - 2) = 0$

$\Rightarrow \quad x + 3 = 0$ or $x - 2 = 0$

$x = -3$ or $x = 2$

When $x = -3$ $\quad y = 2(-3)^3 + 3(-3)^2 - 36(-3) - 12$

$= 69$

When $x = 2$ $\quad y = 2(2)^3 + 3(2)^2 - 36(2) - 12$

$= -56$

$\Rightarrow$ At the points $(-3, 69)$ and $(2, -56)$ the gradient of the curve is 0.

3)

$$\frac{dy}{dx} = 3x^2 + 6x + 21$$

Turning points occur when $\frac{dy}{dx} = 0$

$\Rightarrow \quad 3x^2 + 6x + 21 = 0$

$\div 3 \quad x^2 + 2x + 7 = 0$

$a = 1, \quad b = 2, \quad c = 7$

$b^2 - 4ac = 2^2 - 4 \times 1 \times 7$

$= -24 < 0$

$\Rightarrow$ the equation has no roots

$\Rightarrow$ the curve has no turning points

4)

$y = (x^2 - 5)(x + 3)$

$\Rightarrow \quad y = x^3 + 3x^2 - 5x - 15$

$$\frac{dy}{dx} = 3x^2 + 6x - 5$$

When the gradient = 4

$$\Rightarrow \quad \frac{dy}{dx} = 4$$

$$\Rightarrow \quad 3x^2 + 6x - 5 = 4$$

$$3x^2 + 6x - 9 = 0$$

$$\div 3 \quad x^2 + 2x - 3 = 0$$

$$(x + 3)(x - 1) = 0$$

$$\Rightarrow \quad x + 3 = 0 \quad \text{or} \quad x - 1 = 0$$

$$x = -3 \quad \text{or} \quad x = 1$$

When $x = -3$

$$y = \left((-3)^2 - 5\right)\left((-3) + 3\right)$$

$$= 0$$

When $x = 1$

$$y = \left((1)^2 - 5\right)(1 + 3)$$

$$= -16$$

$\Rightarrow$The gradient of the curve is equal to 4 at the points $(-3, 0)$ and $(1, -16)$

5)

$$\frac{dy}{dx} = 6x + 2$$

Consider the line

$$y = 24 - 10x$$

$$y = -10x + 24$$

$$\text{Gradient of the line } m = -10$$

$$\Rightarrow \quad \frac{dy}{dx} = -10$$

$$\Rightarrow \quad 6x + 2 = -10$$

$$x = -2$$

When $x = -2$

$$y = 3(-2)^2 + 2(-2) + 7$$

$$= 15$$

$\Rightarrow$At the point $(-2, 15)$ the gradient of the tangent to the curve is parallel to the line $y = 24 - 10x$

6)

$$\frac{dy}{dx} = 3x^2 - 12x + 3$$

Consider the line

$$x + 18y = 7$$

$$18y = -x + 7$$

$$y = -\frac{1}{18}x + \frac{7}{18}$$

$$\Rightarrow \quad \text{Gradient of line } m = -\frac{1}{18}$$

$$\text{Gradient of perpendicular line} = 18 \qquad (\text{Using } m_1 m_2 = -1)$$

$$\begin{aligned}
\Rightarrow \qquad 3x^2 - 12x + 3 &= 18 \\
3x^2 - 12x - 15 &= 0 \\
\div 3 \qquad x^2 - 4x - 5 &= 0 \\
(x-5)(x+1) &= 0 \\
\Rightarrow \qquad x - 5 = 0 \quad &\text{or} \quad x + 1 = 0 \\
x = 5 \quad &\text{or} \quad x = -1
\end{aligned}$$

When $x = 5$

$$\begin{aligned}
y &= (5)^3 - 6(5)^2 + 3(5) - 10 \\
&= -20
\end{aligned}$$

When $x = -1$

$$\begin{aligned}
y &= (-1)^3 - 6(-1)^2 + 3(-1) - 10 \\
&= -20
\end{aligned}$$

$\Rightarrow$At the points (5, –20) and (–1, –20) the tangents are perpendicular to the line $x + 18y = 7$

Given that $y = x^3 - 9x^2 - 24x + 12$, find $\frac{dy}{dx}$. [2]

P is the point on the curve where $x = -2$.

(a) Calculate the y co-ordinate of P [2]

(b) Calculate the gradient of the curve at P. [2]

(c) Find the values of x for which the curve has a gradient of –3. [2]

7) $$\frac{dy}{dx} = 3x^2 - 18x - 24$$

(a) When $x = -2$

$$\begin{aligned}
y &= (-2)^3 - 9(-2)^2 - 24(-2) + 12 \\
y &= 16
\end{aligned}$$

(b) Gradient of curve $= \frac{dy}{dx}$

When $x = -2$

$$\begin{aligned}
\frac{dy}{dx} &= 3(-2)^2 - 18(-2) - 24 \\
\Rightarrow \qquad \text{Gradient of curve} &= 24
\end{aligned}$$

(c) When the curve has a gradient $= -3$

$$\begin{aligned}
\frac{dy}{dx} &= -3 \\
\Rightarrow \qquad 3x^2 - 18x - 24 &= -3 \\
\Rightarrow \qquad 3x^2 - 18x - 21 &= 0 \\
\div 3 \qquad x^2 - 6x - 7 &= 0 \\
\Rightarrow \qquad (x+1)(x-7) &= 0 \\
\Rightarrow \qquad x + 1 = 0 \quad &\text{or} \quad x - 7 = 0 \\
x = -1 \quad &\text{or} \quad x = 7
\end{aligned}$$

$\Rightarrow$ The curve has a gradient of –3 when $x = -1$ or $x = 7$

8)

(a)

$$y = x^2(3 - 2x - x^2)$$
$$y = 3x^2 - 2x^3 - x^4$$
$$\frac{dy}{dx} = 6x - 6x^2 - 4x^3$$

(b) Gradient of PQ

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$
$$= \frac{12-0}{-2-1}$$
$$= -4$$

Equation of PQ

$$y - y_1 = m(x - x_1)$$
$$y - 0 = -4(x - 1)$$
$$y = -4x + 4$$

(c) Gradient of curve at P $= 6(-2) - 6(-2)^2 - 4(-2)^3$
$= -4$

This is equal to the gradient of the line PQ

$\Rightarrow$The line PQ is a tangent at P.

Gradient of curve at Q $= 6(1) - 6(1)^2 - 4(1)^3$
$= -4$

$\Rightarrow$The line PQ is a tangent at Q

$\Rightarrow$The line PQ is a tangent to the curve at P and Q

9) $$\frac{dy}{dx} = 3x^2 - 12x + 21$$

If the tangents are perpendicular to the straight line with gradient $-\frac{1}{12}$.

$\Rightarrow$ Gradient of tangent $= 12$ (using $m_1m_2 = -1$)

$\Rightarrow$ $3x^2 - 12x + 21 = 12$

$3x^2 - 12x + 9 = 0$

$\div 3$ $x^2 - 4x + 3 = 0$

$\Rightarrow$ $(x-1)(x-3) = 0$

$\Rightarrow$ $x - 1 = 0$ or $x - 3 = 0$

$x = 1$ or $x = 3$

When $x = 1$

$$y = (1)^3 - 6(1)^2 + 21(1) - 15$$
$$= 1$$

When $x = 3$

$$y = (3)^3 - 6(3)^2 + 21(3) - 15$$
$$= 21$$

$\Rightarrow$The tangents at the points (1, 1) and (3, 21) are each perpendicular to the straight line with gradient $-\frac{1}{12}$

10)

(a)

$$\frac{dy}{dx} = 3x^2 - 8x + 5$$

$$\Rightarrow \quad y = \int \left(3x^2 - 8x + 5\right) dx$$

$$\Rightarrow \quad y = \frac{3x^3}{3} - \frac{8x^2}{2} + 5x + C$$

$$y = x^3 - 4x^2 + 5x + C$$

The curve passes through the point (3, –1)

$$\Rightarrow \quad -1 = (3)^3 - 4(3)^2 + 5(3) + C$$

$$\Rightarrow \quad C = -7$$

$\Rightarrow$ The equation of the curve is

$$y = x^3 - 4x^2 + 5x - 7$$

(b)

$$\text{Gradient of curve} = 1$$

$$\Rightarrow \quad 3x^2 - 8x + 5 = 1$$

$$3x^2 - 8x + 4 = 0$$

$$(3x - 2)(x - 2) = 0$$

$$\Rightarrow \quad 3x - 2 = 0 \quad \text{or} \quad x - 2 = 0$$

$$x = \frac{2}{3} \quad \text{or} \quad x = 2$$

When $x = \frac{2}{3}$

$$y = \left(\frac{2}{3}\right)^3 - 4\left(\frac{2}{3}\right)^2 + 5\left(\frac{2}{3}\right) - 7$$

$$= -5\frac{4}{27}$$

When $x = 2$

$$y = (2)^3 - 4(2)^2 + 5(2) - 7$$

$$= -5$$

$\Rightarrow$ The gradient of the curve is 1 at the points $\left(\frac{2}{3}, -5\frac{4}{27}\right)$ and $(2, -5)$