

- 1
- a e.g.  $a = -2, b = 1 \Rightarrow a^2 - b^2 = 4 - 1 = 3 \Rightarrow a^2 - b^2 > 0$   
and  $a - b = -2 - 1 = -3 \Rightarrow a - b < 0$   
[ any negative value of  $a$  such that  $|a| > |b|$  ]
- b 7 7 is prime and divisible by 7 [ no other examples ]
- c e.g.  $x = \sqrt{2}, y = 2\sqrt{2} \Rightarrow x$  and  $y$  irrational  
and  $xy = 4$  which is rational [ many other examples ]
- d e.g.  $x = -90 \Rightarrow \cos(90 - |x|)^\circ = \cos 0 = 1$   
and  $\sin x^\circ = \sin(-90^\circ) = -1$  [ any  $-ve$   $x$  except multiples of 180 ]
- 2
- a true any number divisible by 6 is also divisible by 2 and  $\therefore$  not prime
- b
- |           |   |    |    |    |     |
|-----------|---|----|----|----|-----|
| $n$       | 1 | 2  | 3  | 4  | 5   |
| $3^n + 2$ | 5 | 11 | 29 | 83 | 245 |
- false e.g.  $n = 5 \Rightarrow 3^n + 2 = 245$  which is divisible by 5 and  $\therefore$  not prime  
[ many other examples ]
- c false e.g.  $n = 4 \Rightarrow \sqrt{n} = 2$  which is rational [ many other examples ]
- d true  $b$  divisible by  $c \Rightarrow b = kc, k \in \mathbb{Z}$   
 $a$  divisible by  $b \Rightarrow a = lb, l \in \mathbb{Z} \Rightarrow a = klc \therefore a$  is divisible by  $c$
- 3
- a assume  $n^3$  odd and  $n$  even, where  $n \in \mathbb{Z}^+$   
 $n$  even  $\Rightarrow n = 2m, m \in \mathbb{Z}$   
 $\Rightarrow n^3 = (2m)^3 = 8m^3 = 2(4m^3)$   
 $4m^3 \in \mathbb{Z} \therefore n^3$  even  
 $\Rightarrow$  contradiction  $\therefore n$  odd
- b assume  $x$  irrational and  $\sqrt{x}$  rational  
 $\sqrt{x}$  rational  $\Rightarrow \sqrt{x} = \frac{p}{q}, p, q \in \mathbb{Z}$   
 $\Rightarrow x = \frac{p^2}{q^2}, p^2, q^2 \in \mathbb{Z} \therefore x$  rational  
 $\Rightarrow$  contradiction  $\therefore \sqrt{x}$  irrational
- c assume  $bc$  not divisible by  $a$  and  $b$  divisible by  $a$  where  $a, b, c \in \mathbb{Z}$   
 $b$  divisible by  $a \Rightarrow b = ka, k \in \mathbb{Z}$   
 $\Rightarrow bc = kac$  which is divisible by  $a$   
 $\Rightarrow$  contradiction  $\therefore b$  is not divisible by  $a$
- d assume  $n^2 - 4n$  odd and  $n$  even, where  $n \in \mathbb{Z}^+$   
 $n$  even  $\Rightarrow n = 2m, m \in \mathbb{Z}$   
 $\Rightarrow n^2 - 4n = (2m)^2 - 4(2m) = 4m^2 - 8m = 2(2m^2 - 4m)$   
 $2m^2 - 4m \in \mathbb{Z} \therefore n^2 - 4n$  even  
 $\Rightarrow$  contradiction  $\therefore n$  odd
- e assume  $m^2 - n^2 = 6$ , where  $m, n \in \mathbb{Z}^+$   
 $m^2 - n^2 = 6 \Rightarrow (m+n)(m-n) = 6$   
 $m, n \in \mathbb{Z}^+ \Rightarrow (m+n), (m-n) \in \mathbb{Z}, (m+n) > (m-n)$  and  $(m+n) > 0$   
 $\therefore m+n = 6$  and  $m-n = 1$  or  $m+n = 3$  and  $m-n = 2$   
adding  $\Rightarrow 2m = 7$  or  $2m = 5$   
 $\Rightarrow m = \frac{7}{2}$  or  $m = \frac{5}{2} \Rightarrow m$  not an integer  
 $\Rightarrow$  contradiction  $\therefore$  no positive integer solutions

- 4**    **a** assume  $x^2 + y^2$  divisible by 4 and  $x, y$  odd integers  
 $x, y$  odd  $\Rightarrow x = 2m + 1, m \in \mathbb{Z}$  and  $y = 2n + 1, n \in \mathbb{Z}$   
 $\Rightarrow x^2 + y^2 = (2m + 1)^2 + (2n + 1)^2$   
 $= 4m^2 + 4m + 1 + 4n^2 + 4n + 1$   
 $= 4(m^2 + m + n^2 + n) + 2$   
 $m^2 + m + n^2 + n \in \mathbb{Z} \therefore x^2 + y^2$  not divisible by 4  
 $\Rightarrow$  contradiction  $\therefore x$  and  $y$  not both odd
- b** assume  $x^2 + y^2$  divisible by 4,  $x$  odd integer and  $y$  even integer  
 $x$  odd,  $y$  even  $\Rightarrow x = 2m + 1, m \in \mathbb{Z}$  and  $y = 2n, n \in \mathbb{Z}$   
 $\Rightarrow x^2 + y^2 = (2m + 1)^2 + (2n)^2$   
 $= 4m^2 + 4m + 1 + 4n^2$   
 $= 4(m^2 + m + n^2) + 1$   
 $m^2 + m + n^2 \in \mathbb{Z} \therefore x^2 + y^2$  not divisible by 4  
 $\Rightarrow$  contradiction  $\therefore x$  odd and  $y$  even not possible  
 same argument applies with  $x$  even and  $y$  odd  
 part **a** shows  $x$  and  $y$  can't both be odd  
 $\therefore x$  and  $y$  both even
- 5**    **a** false e.g.  $a = 2, b = 4 \Rightarrow \log_a b = 2$  which is rational  
 [ many other examples ]
- b** true  $(2n + 1)$  and  $(2n + 3), n \in \mathbb{Z}$  represent any two consecutive odd integers  
 $(2n + 3)^2 - (2n + 1)^2 = 4n^2 + 12n + 9 - (4n^2 + 4n + 1)$   
 $= 8n + 8$   
 $= 8(n + 1)$   
 $n + 1 \in \mathbb{Z} \therefore$  difference is divisible by 8
- c** false e.g.  $n = 13 \Rightarrow n^2 + 3n + 13 = 13(13 + 3 + 1)$  which is divisible by 13  
 [ many other examples ]
- d** true  $x^2 - 2y(x - y) = x^2 - 2xy + 2y^2$   
 $= x^2 - 2xy + y^2 + y^2$   
 $= (x - y)^2 + y^2$   
 for real  $x$  and  $y, (x - y)^2 \geq 0$  and  $y^2 \geq 0 \therefore x^2 - 2y(x - y) \geq 0$
- 6**    **a**  $\sqrt{2} = \frac{p}{q}, p, q \in \mathbb{Z} \Rightarrow 2 = \frac{p^2}{q^2} \Rightarrow p^2 = 2q^2$   
 $\Rightarrow p^2$  even  $\Rightarrow p$  even
- b** assume  $\sqrt{2}$  rational  $\Rightarrow \sqrt{2} = \frac{p}{q}, p, q \in \mathbb{Z}$  and  $p, q$  co-prime  
 part **a**  $\Rightarrow p$  even  $\Rightarrow p = 2n, n \in \mathbb{Z}$   
 $\Rightarrow (2n)^2 = 2q^2$   
 $\Rightarrow q^2 = 2n^2$   
 $\Rightarrow q^2$  even  $\Rightarrow q$  even  
 $\Rightarrow p$  and  $q$  both even  $\therefore$  not co-prime  
 $\Rightarrow$  contradiction  $\therefore \sqrt{2}$  is irrational

- 1    **a** e.g.  $x = \frac{1}{8} \Rightarrow \sqrt[3]{x} = \frac{1}{2}, \frac{1}{2} > \frac{1}{8}$   
       [ any value of  $x$  in the interval  $0 < x < 1$  ]
- b** e.g.  $n = 7 \Rightarrow n^3 - n + 7 = 7(49 - 1 + 1)$  which is divisible by 7  
       [ many other examples ]
- 2    assume  $\sqrt{\pi}$  is rational  $\Rightarrow \sqrt{\pi} = \frac{p}{q}, p, q \in \mathbb{Z}$   
        $\Rightarrow \pi = \frac{p^2}{q^2}, p^2, q^2 \in \mathbb{Z} \therefore \pi$  rational  
        $\Rightarrow$  contradiction  $\therefore \sqrt{\pi}$  irrational
- 3    consider  $15x^2 - 11x + 2 < 0$   
        $\Rightarrow (5x - 2)(3x - 1) < 0$   
        $\Rightarrow \frac{1}{3} < x < \frac{2}{5}$   
       e.g.  $x = 0.35 \Rightarrow 15x^2 - 11x + 2 = -0.0125, -0.0125 < 0$   
       [ any value of  $x$  in the interval  $\frac{1}{3} < x < \frac{2}{5}$  ]
- 4    **a**  $n^2 + 2n = (2m + 1)^2 + 2(2m + 1)$   
        $= 4m^2 + 4m + 1 + 4m + 2$   
        $= 4m^2 + 8m + 3$
- b** assume  $n^2 + 2n$  even and  $n$  odd, where  $n \in \mathbb{Z}$   
        $n$  odd  $\Rightarrow n = 2m + 1, m \in \mathbb{Z}$   
        $\Rightarrow n^2 + 2n = 4m^2 + 8m + 3 = 2(2m^2 + 4m + 1) + 1$   
        $2m^2 + 4m + 1 \in \mathbb{Z} \therefore n^2 + 2n$  odd  
        $\Rightarrow$  contradiction  $\therefore n$  even
- 5    **a**  $k \cos x - \operatorname{cosec} x = 0 \Rightarrow k \cos x = \frac{1}{\sin x}$   
        $\Rightarrow k \sin x \cos x = 1$   
        $\Rightarrow \frac{1}{2} k \sin 2x = 1$   
        $\Rightarrow \sin 2x = \frac{2}{k}$   
        $|\sin 2x| \leq 1 \Rightarrow \left| \frac{2}{k} \right| \leq 1$   
        $\Rightarrow |k| \geq 2$
- b**  $3 \cos x - \operatorname{cosec} x = 0 \Rightarrow \sin 2x = \frac{2}{3}$   
        $2x = 41.810, 180 - 41.810, 360 + 41.810, 540 - 41.810$   
        $2x = 41.810, 138.190, 401.810, 498.190$   
        $x = 20.9, 69.1, 200.9, 249.1$  (1dp)

- 6 assume  $x^2 - y^2 = 1$ , where  $x, y \in \mathbb{Z}^+$   
 $x^2 - y^2 = 1 \Rightarrow (x+y)(x-y) = 1$   
 $x, y \in \mathbb{Z}^+ \Rightarrow (x+y), (x-y) \in \mathbb{Z}$  and  $(x+y) > 0$   
 $\therefore x+y = 1$  and  $x-y = 1$   
 adding  $\Rightarrow 2x = 2$   
 $\Rightarrow x = 1$   
 $\Rightarrow y = 0$   
 $\Rightarrow$  contradiction  $\therefore$  no positive integer solutions
- 7 a false e.g.  $a = \sqrt{2}, b = 1 - \sqrt{2} \Rightarrow a$  and  $b$  irrational  
 and  $a + b = 1$  which is rational  
 [ many other examples ]
- b true  $m, n$  consecutive odd integers  $\Rightarrow m = 2a + 1, n = 2a + 3, a \in \mathbb{Z}$   
 $\Rightarrow m + n = 2a + 1 + 2a + 3 = 4a + 4 = 4(a + 1)$   
 $a + 1 \in \mathbb{Z} \therefore m + n$  divisible by 4
- c false e.g.  $x = \frac{5\pi}{3} \Rightarrow \cos x = \frac{1}{2}$  and  $1 + \sin x = 1 - \frac{\sqrt{3}}{2}, \frac{1}{2} > 1 - \frac{\sqrt{3}}{2}$   
 [ any value of  $x$  of the form  $2n\pi + y, n \in \mathbb{Z}, -\frac{\pi}{2} < y < 0$  ]
- 8 a  $\log_2 3 = \frac{p}{q} \Rightarrow 2^{\frac{p}{q}} = 3$   
 $\Rightarrow (2^{\frac{p}{q}})^q = 3^q$   
 $\Rightarrow 2^p = 3^q$
- b assume  $\log_2 3$  is rational  $\Rightarrow \log_2 3 = \frac{p}{q}, p, q \in \mathbb{Z}, q \neq 0$   
 $\Rightarrow 2^p = 3^q$   
 2 and 3 are co-prime  $\Rightarrow p = q = 0$   
 $\Rightarrow$  contradiction  $\therefore \log_2 3$  is irrational
- c e.g.  $a = 2, b = \sqrt{2} \Rightarrow a$  rational and  $b$  irrational  
 and  $\log_a b = \frac{1}{2}$  which is rational  
 [ many other examples ]
- 9 a  $y = \frac{x-2}{4x}$  swap  $x = \frac{y-2}{4y}$   
 $4xy = y - 2$   
 $y(4x - 1) = -2$   
 $y = \frac{2}{1-4x}$   
 $f^{-1}(x) = \frac{2}{1-4x}, x \in \mathbb{R}, x \neq \frac{1}{4}$
- b  $f(x) = f^{-1}(x) \Rightarrow \frac{x-2}{4x} = \frac{2}{1-4x}$   
 $\Rightarrow (x-2)(1-4x) = 8x$   
 $\Rightarrow 4x^2 - x + 2 = 0$   
 $b^2 - 4ac = 1 - 32 = -31$   
 $b^2 - 4ac < 0 \Rightarrow$  no real roots  
 $\therefore$  no real values of  $x$  for which  $f(x) = f^{-1}(x)$