1 **a** $\overrightarrow{AB} = (5\mathbf{i} - 6\mathbf{k}) - (\mathbf{i} + 6\mathbf{j} + 4\mathbf{k})$
$= 4\mathbf{i} - 6\mathbf{j} - 10\mathbf{k}$
$\therefore\ \mathbf{r} = \mathbf{i} + 6\mathbf{j} + 4\mathbf{k} + \lambda(4\mathbf{i} - 6\mathbf{j} - 10\mathbf{k})$

b $1 + 4\lambda = 5 + \mu \quad (1)$
$6 - 6\lambda = -5 - 4\mu \quad (2)$
$4\times(1) + (2) \Rightarrow 10 + 10\lambda = 15$
$\lambda = \frac{1}{2}$
$\therefore$ pos. vector of $C = 3\mathbf{i} + 3\mathbf{j} - \mathbf{k}$

c pos. vector of mid-point of AB
$= \overrightarrow{OA} + \frac{1}{2}\overrightarrow{AB}$
$= (\mathbf{i} + 6\mathbf{j} + 4\mathbf{k}) + \frac{1}{2}(4\mathbf{i} - 6\mathbf{j} - 10\mathbf{k})$
$= 3\mathbf{i} + 3\mathbf{j} - \mathbf{k}$
$\therefore$ C is mid-point of AB

2 **a** $\overrightarrow{PQ} = (3\mathbf{i} + \mathbf{j}) - (5\mathbf{i} - 2\mathbf{j} + 2\mathbf{k})$
$= -2\mathbf{i} + 3\mathbf{j} - 2\mathbf{k}$
$\therefore\ \mathbf{r} = 5\mathbf{i} - 2\mathbf{j} + 2\mathbf{k} + \lambda(-2\mathbf{i} + 3\mathbf{j} - 2\mathbf{k})$

b $5 - 2\lambda = 4 + 5\mu \quad (1)$
$-2 + 3\lambda = 6 - \mu \quad (2)$
$2 - 2\lambda = -1 + 3\mu \quad (3)$
$(1) - (3) \Rightarrow 3 = 5 + 2\mu$
$\mu = -1,\ \lambda = 3$
check (2) $\quad -2 + 3(3) = 6 - (-1)$
true $\therefore$ intersect
pos. vector of int. $= -\mathbf{i} + 7\mathbf{j} - 4\mathbf{k}$

c $|-2\mathbf{i} + 3\mathbf{j} - 2\mathbf{k}| = \sqrt{4+9+4} = \sqrt{17}$
$|5\mathbf{i} - \mathbf{j} + 3\mathbf{k}| = \sqrt{25+1+9} = \sqrt{35}$
$(-2\mathbf{i} + 3\mathbf{j} - 2\mathbf{k}).(5\mathbf{i} - \mathbf{j} + 3\mathbf{k})$
$= -10 - 3 - 6 = -19$
$\theta = \cos^{-1}\left|\frac{-19}{\sqrt{17}\sqrt{35}}\right| = 38.8°$

3 **a** $5 + 2\lambda = 7 - \mu \quad (1)$
$-\lambda = -3 + \mu \quad (2)$
$1 + 2\lambda = 7 - 2\mu \quad (3)$
$(1) + (2) \Rightarrow 5 + \lambda = 4$
$\lambda = -1,\ \mu = 4$
check (3) $\quad 1 + 2(-1) = 7 - 2(4)$
true $\therefore$ intersect
pos. vector of int. $= 3\mathbf{i} + \mathbf{j} - \mathbf{k}$

b diagonals bisect each other
let M be point of intersection
$\therefore\ \overrightarrow{AM} = (3\mathbf{i} + \mathbf{j} - \mathbf{k}) - (9\mathbf{i} - 2\mathbf{j} + 5\mathbf{k})$
$= -6\mathbf{i} + 3\mathbf{j} - 6\mathbf{k}$
$\overrightarrow{OC} = \overrightarrow{OA} + 2\overrightarrow{AM}$
$= (9\mathbf{i} - 2\mathbf{j} + 5\mathbf{k}) + 2(-6\mathbf{i} + 3\mathbf{j} - 6\mathbf{k})$
$= -3\mathbf{i} + 4\mathbf{j} - 7\mathbf{k}$

c area of triangle $ABC = \frac{1}{2} \times 54 = 27$
$\overrightarrow{AC} = 2(-6\mathbf{i} + 3\mathbf{j} - 6\mathbf{k}) = 6(-2\mathbf{i} + \mathbf{j} - 2\mathbf{k})$
$|\overrightarrow{AC}| = 6\sqrt{4+1+4} = 18$
let distance of B from $l_1 = d$
$\therefore\ \frac{1}{2} \times 18 \times d = 27$
$d = 3$

4 **a** $\overrightarrow{AB} = (2\mathbf{i} - \mathbf{j} + 2\mathbf{k}) - (4\mathbf{i} + 2\mathbf{j} - 4\mathbf{k})$
$= -2\mathbf{i} - 3\mathbf{j} + 6\mathbf{k}$
$\therefore\ \mathbf{r} = 4\mathbf{i} + 2\mathbf{j} - 4\mathbf{k} + \lambda(-2\mathbf{i} - 3\mathbf{j} + 6\mathbf{k})$

b $\mathbf{r} = 4\mathbf{i} - 7\mathbf{j} - \mathbf{k} + \mu(6\mathbf{j} - 2\mathbf{k})$

c $-7 + 6\mu = 2 \Rightarrow \mu = \frac{3}{2}$
sub. $\mu = \frac{3}{2}$ in l_2
$\mathbf{r} = 4\mathbf{i} - 7\mathbf{j} - \mathbf{k} + \frac{3}{2}(6\mathbf{j} - 2\mathbf{k})$
$= 4\mathbf{i} + 2\mathbf{j} - 4\mathbf{k}$ $\therefore$ A lies on l_2

d $|-2\mathbf{i} - 3\mathbf{j} + 6\mathbf{k}| = \sqrt{4+9+36} = 7$
$|6\mathbf{j} - 2\mathbf{k}| = \sqrt{36+4} = \sqrt{40}$
$(-2\mathbf{i} - 3\mathbf{j} + 6\mathbf{k}).(6\mathbf{j} - 2\mathbf{k})$
$= 0 - 18 - 12 = -30$
$\theta = \cos^{-1}\left|\frac{-30}{7\sqrt{40}}\right| = 47.3°$ (1dp)

5 **a** $\overrightarrow{AB} = (4\mathbf{i} + \mathbf{j} - 8\mathbf{k}) - (5\mathbf{i} - \mathbf{j} - 10\mathbf{k})$
$= -\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$
$\therefore\ \mathbf{r} = 5\mathbf{i} - \mathbf{j} - 10\mathbf{k} + \lambda(-\mathbf{i} + 2\mathbf{j} + 2\mathbf{k})$

b $5 - \lambda = 0 \Rightarrow \lambda = 5$
sub. $\lambda = 5$ in l
$\mathbf{r} = 9\mathbf{j}\ \therefore\ C\,(0, 9, 0)$

c $\overrightarrow{OD} = (5-\lambda)\mathbf{i} + (-1+2\lambda)\mathbf{j} + (-10+2\lambda)\mathbf{k}$
$\overrightarrow{OD}.(-\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}) = 0$
$-(5 - \lambda) + 2(-1 + 2\lambda) + 2(-10 + 2\lambda) = 0$
$9\lambda - 27 = 0$
$\lambda = 3,\ \overrightarrow{OD} = 2\mathbf{i} + 5\mathbf{j} - 4\mathbf{k}$
$\therefore\ D\,(2, 5, -4)$

d $OD = \sqrt{4+25+16} = \sqrt{45} = 3\sqrt{5}$
$CD = \sqrt{4+16+16} = 6$
area $= \frac{1}{2} \times 6 \times 3\sqrt{5} = 9\sqrt{5}$

6 **a** $-6 + 4\lambda = 6 \Rightarrow \lambda = 3$
sub. $\lambda = 3$ in l_1
$\mathbf{r} = (\mathbf{i} - 6\mathbf{j} - 2\mathbf{k}) + 3(4\mathbf{j} - \mathbf{k})$
$= \mathbf{i} + 6\mathbf{j} - 5\mathbf{k}$
$\therefore\ P$ lies on l_1

b $1 = 4 + 3\mu \quad \Rightarrow \mu = -1$
sub. $\mu = -1$ in l_2
$\mathbf{r} = 4\mathbf{i} + \mathbf{j} + \mathbf{k} - (\mathbf{i} + \mathbf{j} + \mathbf{k})$
$\overrightarrow{OQ} = \mathbf{i} - 2\mathbf{j} - 3\mathbf{k}$

c $PQ = \sqrt{0+64+4} = \sqrt{68} = 2\sqrt{17}$
$|3\mathbf{i} - 2\mathbf{j} + 2\mathbf{k}| = \sqrt{9+4+4} = \sqrt{17}$
$\therefore\ \overrightarrow{OR} = \overrightarrow{OQ} \pm 2(3\mathbf{i} - 2\mathbf{j} + 2\mathbf{k})$
$= (-5\mathbf{i} + 2\mathbf{j} - 7\mathbf{k})$ or $(7\mathbf{i} - 6\mathbf{j} + \mathbf{k})$

7 **a** $\overrightarrow{AB} = (4\mathbf{i} + 6\mathbf{j} + 2\mathbf{k}) - (4\mathbf{i} + 5\mathbf{j} + 6\mathbf{k})$
$= \mathbf{j} - 4\mathbf{k}$
$\therefore\ \mathbf{r} = 4\mathbf{i} + 5\mathbf{j} + 6\mathbf{k} + \lambda(\mathbf{j} - 4\mathbf{k})$

b $4 = 1 + \mu$ (1)
$5 + \lambda = 5 + \mu$ (2)
$6 - 4\lambda = -3 - \mu$ (3)
(1) $\Rightarrow$ $\mu = 3$
sub. (2) $\Rightarrow$ $\lambda = 3$
check (3) $6 - 4(3) = -3 - (3)$
true $\therefore$ intersect
pos. vector of int. $= 4\mathbf{i} + 8\mathbf{j} - 6\mathbf{k}$

c $|(\mathbf{j} - 4\mathbf{k})| = \sqrt{1+16} = \sqrt{17}$
$|(\mathbf{i} + \mathbf{j} - \mathbf{k})| = \sqrt{1+1+1} = \sqrt{3}$
$(\mathbf{j} - 4\mathbf{k}).(\mathbf{i} + \mathbf{j} - \mathbf{k}) = 0 + 1 + 4 = 5$
$\theta = \cos^{-1}\left|\frac{5}{\sqrt{3}\sqrt{17}}\right| = 45.6°$ (1dp)

d let closest point be C
$\overrightarrow{OC} = (1 + \mu)\mathbf{i} + (5 + \mu)\mathbf{j} + (-3 - \mu)\mathbf{k}$
$\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA}$
$= (-3 + \mu)\mathbf{i} + \mu\mathbf{j} + (-9 - \mu)\mathbf{k}$
AC must be perpendicular to l_2
$\therefore\ \overrightarrow{AC}.(\mathbf{i} + \mathbf{j} - \mathbf{k}) = 0$
$(-3 + \mu) + \mu - (-9 - \mu) = 0$
$\mu = -2$
$\therefore\ \overrightarrow{OC} = -\mathbf{i} + 3\mathbf{j} - \mathbf{k}$

8 **a** $\mathbf{r} = \mathbf{i} + 12\mathbf{j} + 3\mathbf{k} + \lambda(4\mathbf{i} - 7\mathbf{j} - \mathbf{k})$
$\mathbf{r} = 18\mathbf{i} - 20\mathbf{j} + 4\mathbf{k} + \mu(-3\mathbf{i} + 6\mathbf{j} - \mathbf{k})$

b $1 + 4\lambda = 18 - 3\mu$ (1)
$12 - 7\lambda = -20 + 6\mu$ (2)
$3 - \lambda = 4 - \mu$ (3)
(1) + 4×(3) $\Rightarrow$ $13 = 34 - 7\mu$
$\mu = 3,\ \lambda = 2$
check (2) $12 - 7(2) = -20 + 6(3)$
true $\therefore$ intersect
pos. vector of int. $= 9\mathbf{i} - 2\mathbf{j} + \mathbf{k}$

c length of actual tunnel
$= |2(4\mathbf{i} - 7\mathbf{j} - \mathbf{k})| + |3(-3\mathbf{i} + 6\mathbf{j} - \mathbf{k})|$
$= 2\sqrt{16+49+1} + 3\sqrt{9+36+1}$
$= 2\sqrt{66} + 3\sqrt{46}$
vector between ends
$= (18\mathbf{i} - 20\mathbf{j} + 4\mathbf{k}) - (\mathbf{i} + 12\mathbf{j} + 3\mathbf{k})$
$= 17\mathbf{i} - 32\mathbf{j} + \mathbf{k}$
length of direct tunnel
$= \sqrt{289+1024+1}$
$= \sqrt{1314}$
direct tunnel shorter by
$2\sqrt{66} + 3\sqrt{46} - \sqrt{1314}$
$= 0.3459$ units
$= 20 \times 0.3459$ m
$= 6.92$ m (3sf)