C3 NUMERICAL METHODS Worksheet A

1 Show in each case that there is a root of the equation $f(x) = 0$ in the given interval.

a $f(x) = x^3 + 3x - 7$ $(1, 2)$

b $f(x) = 5\cos x - 3x$ $(0.5, 1)$

c $f(x) = 2e^x + x + 5$ $(-6, -5)$

d $f(x) = x^4 - 5x^2 + 1$ $(2.1, 2.2)$

e $f(x) = \ln(4x - 1) + x^2$ $(0.4, 0.5)$

f $f(x) = e^{-x} - 9\cos 4x$ $(10, 11)$

2 Given that $|N| \le 5$, find in each case the integer N such that there is a root of the equation $f(x) = 0$ in the interval $(N, N + 1)$.

a $f(x) = x^3 - 3\sqrt{x} - 4$

b $f(x) = x\ln x - \frac{12}{x}$

c $f(x) = 2x^5 + 4x + 15$

d $f(x) = e^{x-1} + 4x - 2$

e $f(x) = e^x - 3\sin x$

f $f(x) = \tan(0.1x) + x - 6$

3 Show in each case that there is a root of the given equation in the given interval.

a $x^3 = 12 - \frac{x}{4}$ $[2, 3]$

b $12e^x = 9 - 4x$ $[-1, 0]$

c $10\ln 3x = 5 - 7x^2$ $[0.47, 0.48]$

d $\sin 4x = 7e^x$ $[-6.5, -6]$

e $4^x = 3x + 10$ $[-4, -3]$

f $\tan(\frac{1}{2}x) = 2x - 1$ $[2.6, 2.7]$

4 In each case there is a root of the equation $f(x) = 0$ in the given interval.
Find the integer, a, such that this root lies in the interval $(\frac{a}{10}, \frac{a+1}{10})$.

a $f(x) = x^4 + \frac{3}{x} - 5$ $(1, 2)$

b $f(x) = x - \ln(6 + x^2)$ $(2, 3)$

c $f(x) = 5x^3 - 3x^2 + 11$ $(-2, -1)$

d $f(x) = \frac{8}{x} - \cos x$ $(11, 12)$

e $f(x) = \operatorname{cosec} x + \sqrt{x}$ $(5, 6)$

f $f(x) = x^2 - 7e^{2x+5}$ $(-3, -2)$

5 **a** On the same set of axes, sketch the graphs of $y = x^3$ and $y = 4 - x$.

b Hence, show that the equation $x^3 + x - 4 = 0$ has exactly one real root.

c Show that this root lies in the interval $(1, 1.5)$.

6 $$f : x \to x\ln x - 1, \ x \in \mathbb{R}, \ x > 0.$$

a On the same set of axes, sketch the curves $y = \ln x$ and $y = \frac{1}{x}$.

b Hence show that the equation $f(x) = 0$ has exactly one real root.

The real root of $f(x) = 0$ is α.

c Find the integer n such that $n < \alpha < n + 1$.

7 **a** On the same set of axes, sketch the curves $y = e^x$ and $y = 5 - x^2$.

b Hence show that the equation $e^x + x^2 - 5 = 0$ has exactly one negative and one positive real root.

c Show that the negative root lies in the interval $(-3, -2)$.

The positive root, α, is such that $\frac{n}{10} < \alpha < \frac{n+1}{10}$, where n is an integer.

d Find the value of n.

1 For each equation, show that it can be rearranged into the given iterative form. Use this and the given value of x_0 to find x_1, x_2 and x_3. Give your value of x_3 correct to 4 decimal places.

a	$9 + 4x - 2x^3 = 0$	$x_{n+1} = \sqrt[3]{2x_n + 4.5}$	$x_0 = 2$
b	$e^x - 8x + 5 = 0$	$x_{n+1} = \ln(8x_n - 5)$	$x_0 = 3$
c	$\tan x - 5x + 13 = 0$	$x_{n+1} = \arctan(5x_n - 13)$	$x_0 = -1.2$
d	$\ln x + \sqrt{x} + 1.4 = 0$	$x_{n+1} = e^{-(\sqrt{x_n} + 1.4)}$	$x_0 = 0.16$

2 For each equation, show that it can be rearranged into the given iterative form and state the values of the constants a and b. Use this and the given value of x_0 to find x_1, x_2 and x_3. Give your value of x_3 correct to 3 decimal places.

a	$e^{2x-1} - 6x = 0$	$x_{n+1} = a(\ln bx_n + 1)$	$x_0 = 1.7$
b	$\frac{2}{x} + \cos x - 3 = 0$	$x_{n+1} = \frac{a}{b - \cos x_n}$	$x_0 = 0.8$
c	$2x^3 - 6x - 11 = 0$	$x_{n+1} = \sqrt{a + \frac{b}{x_n}}$	$x_0 = 2$
d	$15\ln(x + 3) - 4x = 0$	$x_{n+1} = e^{ax_n} + b$	$x_0 = -2.5$

3 In each case, use the given iteration formula and value of x_0 to find a root of the equation $f(x) = 0$ to the stated degree of accuracy. Justify the accuracy of your answers.

a	$f(x) = 10^x + 3x - 4$	$x_{n+1} = \log_{10}(4 - 3x_n)$	$x_0 = 0.44$	3 decimal places
b	$f(x) = x^2 + \frac{1}{x - 5}$	$x_{n+1} = \sqrt{\frac{x_n^3 + 1}{5}}$	$x_0 = 0.5$	2 significant figures
c	$f(x) = 30 - 5x + \sin 2x$	$x_{n+1} = 6 + 0.2\sin 2x_n$	$x_0 = 6$	3 significant figures
d	$f(x) = e^{4-x} - \ln x$	$x_{n+1} = 4 - \ln(\ln x_n)$	$x_0 = 3.7$	3 decimal places

4 $$f(x) = x^5 - 10x^3 + 4.$$

The equation $f(x) = 0$ has a root in the interval $-4 < x < -3$.

a Use the iteration formula $x_{n+1} = \sqrt[5]{10x_n^3 - 4}$ and the starting value $x_0 = -3.2$ to find the value of this root correct to 2 decimal places.

The equation $f(x) = 0$ can be rearranged into the iterative form $x_{n+1} = \sqrt[3]{\frac{a}{b - x_n^2}}$.

b Find the values of the constants a and b in this formula.

The equation $f(x) = 0$ has another root in the interval $0 < x < 1$.

c Using the iteration formula with your values from part **b** and the starting value $x_0 = 1$, find the value of this root correct to 3 decimal places.

5 $$f : x \to \arcsin 2x - 0.5x - 0.7,\ x \in \mathbb{R},\ |x| \le 0.5$$

The equation $f(x) = 0$ can be rearranged into the iterative form $x_{n+1} = a\sin(bx_n + c)$.

a Find the values of the constants a, b and c in this formula.

The equation $f(x) = 0$ has a solution in the interval (0.3, 0.4).

b Using the iterative formula with your values from part **a** and the starting value $x_0 = 0.4$, find this solution correct to 3 decimal places.

1 **a** Show that the equation $x^3 - 7x - 11 = 0$ has a real root in the interval (3, 4).

b Using the iterative formula $x_{n+1} = \sqrt{7 + \frac{11}{x_n}}$, with $x_0 = 3.2$, find x_1, x_2 and x_3, giving the value of x_3 correct to 2 decimal places.

2 $$f(x) \equiv 4 \operatorname{cosec} x - 5 + 2x.$$

a Find the values of f(4) and f(5).

b Hence show that the equation $f(x) = 0$ has a root in the interval (4, 5).

The iterative formula $x_{n+1} = a + \dfrac{b}{\sin x_n}$, where a and b are constants, is used to find this root.

c Find the values of a and b.

d Starting with $x_0 = 4.5$, use the iterative formula with your values of a and b to find 3 further approximations of the root, giving your final answer correct to 3 decimal places.

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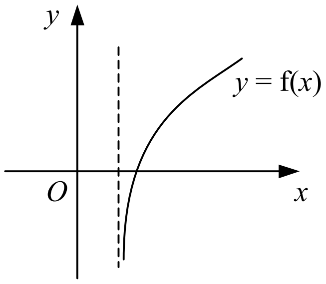


The diagram shows the curve with equation $y = f(x)$ where

$$f : x \to 2x + \ln(3x - 1), \ x \in \mathbb{R}, \ x > \tfrac{1}{3}.$$

Given that $f(\alpha) = 0$,

a show that $0.4 < \alpha < 0.5$,

b use the iterative formula $x_{n+1} = \frac{1}{3}(1 + e^{-2x_n})$, with $x_0 = 0.45$, to find the value of α correct to 3 decimal places.

4 **a** On the same set of axes, sketch the curves $y = \cos x$ and $y = x^2$ for $-\frac{\pi}{2} < x < \frac{\pi}{2}$.

b Show that the equation $\cos x - x^2 = 0$ has exactly one positive and one negative real root.

c Show that the positive real root lies in the interval [0.8, 0.9].

d Use the iteration formula $x_{n+1} = \sqrt{\cos x_n}$ and the starting value $x_0 = 0.8$ to find the positive root correct to 2 decimal places.

5 $$f(x) \equiv e^{5-2x} - x^5.$$

Show that the equation $f(x) = 0$

a has a root in the interval (1.4, 1.5),

b can be written as $x = e^{1-kx}$, stating the value of k.

c Using the iteration formula $x_{n+1} = e^{1-kx_n}$, with $x_0 = 1.5$ and the value of k found in part **b**, find x_1, x_2 and x_3. Give the value of x_3 correct to 3 decimal places.

6 $$f : x \to 2^x + x^3 - 5,\ x \in \mathbb{R}.$$

a Show that there is a solution of the equation $f(x) = 0$ in the interval $1.3 < x < 1.4$

b Using the iterative formula $x_{n+1} = \sqrt[3]{5 - 2^{x_n}}$, with $x_0 = 1.4$, find x_1, x_2, x_3 and x_4.

c Hence write down an approximation for this solution of the equation $f(x) = 0$ to an appropriate degree of accuracy.

Another attempt is made to find the solution using the iterative formula $x_{n+1} = \dfrac{\ln(5 - x_n^3)}{\ln 2}$.

d Describe the outcome of this attempt.

7 $$f(x) = 2x^3 + 4x - 9.$$

a Find $f'(x)$.

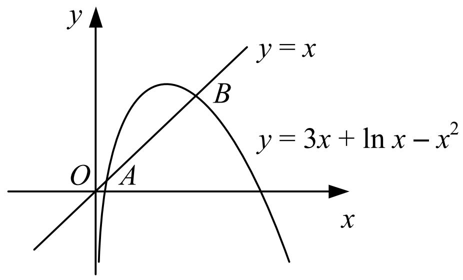
b Hence show that the equation $f(x) = 0$ has exactly one real root.

c Show that this root lies in the interval $(1.2, 1.3)$.

d Use the iterative formula $x_{n+1} = \sqrt[3]{4.5 - 2x_n}$, with $x_0 = 1.2$, to find the root of $f(x) = 0$ correct to 2 decimal places.

e Justify the accuracy of your answer.

8



The diagram shows part of the curve with equation $y = 3x + \ln x - x^2$ and the line $y = x$.

Given that the curve and line intersect at the points A and B, show that

a the x-coordinates of A and B are the solutions of the equation $x = e^{x^2 - 2x}$,

b the x-coordinate of A lies in the interval $(0.4, 0.5)$,

c the x-coordinate of B lies in the interval $(2.3, 2.4)$.

d Use the iteration formula $x_{n+1} = e^{x_n^2 - 2x_n}$, with $x_0 = 0.5$, to find the x-coordinate of A correct to 2 decimal places.

e Justify the accuracy of your answer to part **d**.

9 **a** On the same set of axes, sketch the graphs of $y = x^4$ and $y = 5x + 2$.

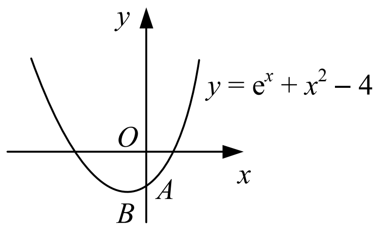
b Show that the equation $x^4 - 5x - 2 = 0$ has exactly one positive and one negative real root.

c Use the iteration formula $x_{n+1} = \sqrt[4]{5x_n + 2}$, with $x_0 = 1.8$, to find x_1, x_2, x_3 and x_4, giving the value of x_4 correct to 3 decimal places.

d Show that the equation $x^4 - 5x - 2 = 0$ can be written in the form $x = \dfrac{a}{x^3 + b}$, stating the values of a and b.

e Use the iteration formula $x_{n+1} = \dfrac{a}{x_n^3 + b}$, with $x_0 = -0.4$ and your values of a and b, to find the negative real root of the equation correct to 4 decimal places.

1



The diagram shows the curve $y = e^x + x^2 - 4$. The curve intersects the y-axis at the point A and has a stationary point at B.

a Find $\frac{dy}{dx}$. **(1)**

b Find an equation for the tangent to the curve at A. **(2)**

c Show that the x-coordinate of B lies in the interval $[-0.4, -0.3]$. **(3)**

d Using the iteration formula $x_{n+1} = \frac{1}{3}(x_n - e^{x_n})$, with $x_0 = -0.3$, find the x-coordinate of B correct to 3 decimal places. **(4)**

2 The function f is defined by

$$f(x) \equiv \sin(x - 6) - \ln(x^2 + 1), \quad x \in \mathbb{R},$$

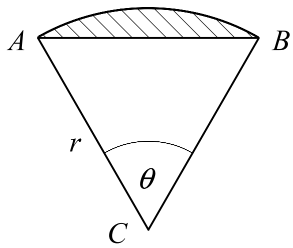
where x is measured in radians.

The equation $f(x) = 0$ has a root in the interval $k < x < k + 1$, where k is a positive integer.

a Find the value of k. **(3)**

b Use the iteration formula $x_{n+1} = \sqrt{e^{\sin(x_n - 6)} - 1}$, with $x_0 = k$, to find three further approximations for this root, giving your answers to 4 decimal places. **(3)**

3



The diagram shows a sector ABC of a circle, centre C, radius r. Angle ACB is θ radians.

Given that the ratio of the area of the shaded segment to the area of triangle ABC is $1 : 4$,

a show that $4\theta - 5\sin\theta = 0$, **(4)**

b use the iterative formula $\theta_{n+1} = \frac{5}{4}\sin\theta_n$, with $\theta_0 = 1.1$, to find the value of θ correct to 2 decimal places. **(4)**

4 $$f : x \to e^{x^2} - x - 3, \quad x \in \mathbb{R}.$$

The equation $f(x) = 0$ can be rearranged into the iterative form $x_{n+1} = \sqrt{\ln(ax_n + b)}$.

a Find the values of the constants a and b in this formula. **(3)**

The equation $f(x) = 0$ has a solution in the interval $(1, 2)$.

b Using the iterative formula with your values from part **a** and a suitable starting value, find this solution correct to 3 decimal places. **(4)**

5 $\mathrm{f} : x \to x^2 - 9,\ x \in \mathbb{R},\ x \geq 0,$

$\mathrm{g} : x \to x^3,\ x \in \mathbb{R}.$

a Find $\mathrm{f}^{-1}(x)$ and state its domain and range. **(4)**

b On the same set of axes, sketch the curves $y = \mathrm{f}(x)$ and $y = \mathrm{f}^{-1}(x)$. **(2)**

c Show that the equation $\mathrm{f}^{-1}(x) + \mathrm{g}(x) = 0$ has a root in the interval $[-2, -1]$. **(3)**

d Use the iterative formula $x_{n+1} = -(x_n + 9)^{\frac{1}{6}}$, with $x_0 = -1$, to find this root correct to 3 decimal places. **(4)**

6 **a** On the same diagram, sketch the curves $y = \frac{1}{x}$ and $y = |-x^2 - 3x|$, showing the coordinates of any points of intersection with the coordinate axes. **(3)**

The curves intersect at the point P.

b Show that the x-coordinate of P can be found by solving the equation $x^3 + 3x^2 - 1 = 0$. **(3)**

c Use the iteration formula $x_{n+1} = \frac{1}{\sqrt{x_n + 3}}$, with $x_0 = 0$, to find the x-coordinate of P correct to 3 decimal places. **(4)**

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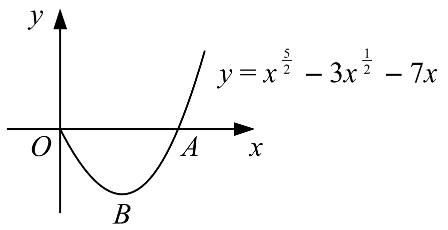

The diagram shows the curve $y = x^{\frac{5}{2}} - 3x^{\frac{1}{2}} - 7x,\ x \geq 0,$ which crosses the x-axis at the point A, where $x = \alpha$, and has a stationary point at B, where $x = \beta$.

Show that

a $4 < \alpha < 5$, **(2)**

b $2 < \beta < 3$, **(4)**

c $x = \beta$ is a solution to the equation $x = \sqrt{0.6 + 2.8x^{\frac{1}{2}}}$. **(3)**

d Use the iterative formula $x_{n+1} = \sqrt{0.6 + 2.8x_n^{\frac{1}{2}}}$, with $x_0 = 2.1$, to find β correct to 4 significant figures. **(4)**

8 The curve with equation $y = 3x - \ln x$ passes through the point $P\,(1, 3)$.

a Find an equation for the normal to the curve at P. **(4)**

The normal to the curve at P intersects the curve again at the point Q.

b Show that the x-coordinate of Q satisfies the equation

$$2 \ln x - 7x + 7 = 0.$$ **(1)**

The x-coordinate of Q is to be found using an iteration of the form $x_{n+1} = \mathrm{e}^{k(x_n - 1)}$.

c Find the value of the constant k. **(2)**

d Using $x_0 = 0.5$, find the x-coordinate of Q correct to 3 decimal places. **(4)**

e Justify the accuracy of your answer to part **d**. **(2)**