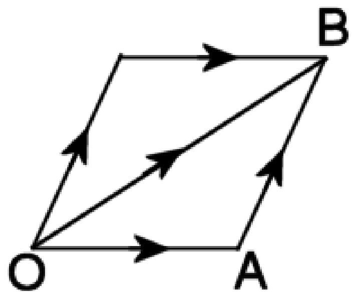
Statics

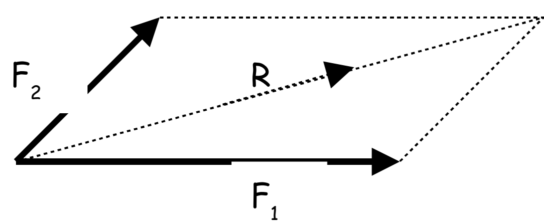
Statics is the study of stationary objects. We will consider a variety of situations where bodies are acted upon by a number of forces. A few of the concepts introduced in our work on vectors will be built upon in this unit.

Resolving Forces

In the vectors unit we were made aware of the fact that the resultant of two vectors is the diagonal of a parallelogram, as highlighted in the diagram below.

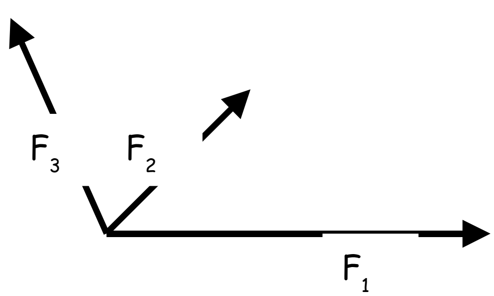


This idea can be applied to forces:



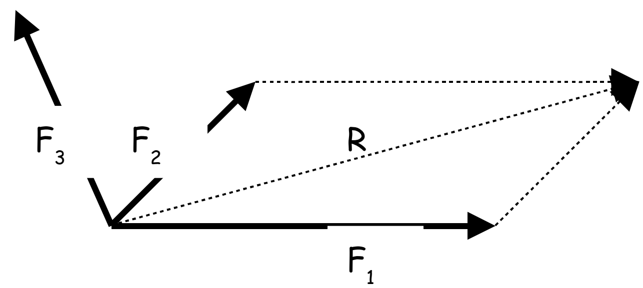
In a real world sense the path R is the direction that a particle would take if it were to be acted upon by the forces F_1 and F_2. This principle can be applied to more than two forces.

Suppose that a particle is acted upon by the forces F_1, F_2 and F_3.

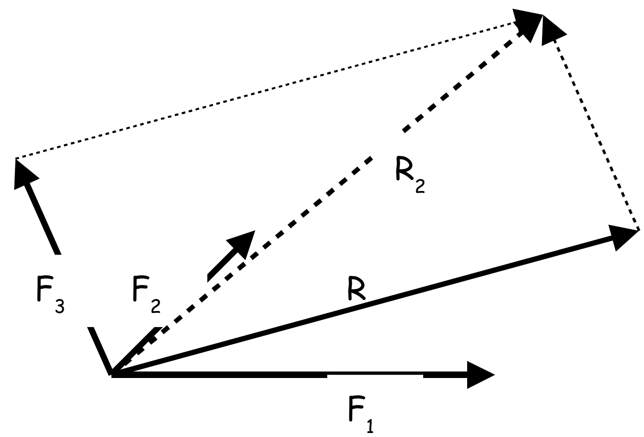


The diagrams below should explain the path that the particle will follow.

Firstly find the resultant of the forces F_1, and F_2 to give R.



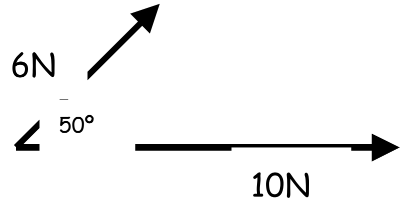
Second find the resultant of R and F_3 .



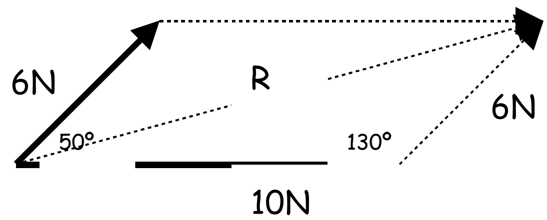
The path R_2 shows the direction of motion of the particle as it is the resultant of the three forces F_1, F_2 and F_3.

<u>Example 1</u>

Two forces act on a particle as outlined in the diagram below. Find the resultant force acting on the object and the angle it makes with the 10N force.



We need to find the magnitude and direction of the force R.



Using Cosine Rule:

$$R^2 = 6^2 + 10^2 - 2 \times 6 \times 10\cos 130$$

$$R = \sqrt{213.1}$$

Using Sine rule to find the angle that the resultant R makes with the 10N force:

$$\frac{\text{Sin } 130}{R} = \frac{\text{Sin } \theta}{6}$$

$$\text{Sin } \theta = \frac{6\text{Sin } 130}{\sqrt{213.1}}$$

$$\theta = 18.4^\circ$$

Later work will involve more than two forces but the method used is the one we introduced in the vectors unit (resolving into **i** and **j** components).

Questions involving forces will also be given in **i**, **j** notation.

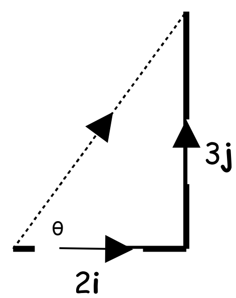
Example 2

Forces E, F and G are applied to a particle. Find the resultant of the three forces in terms of **i** and **j** components. Find the magnitude and the direction of the resulting force.

$E = (7\mathbf{i} - 3\mathbf{j})$N, $F = (-3\mathbf{i} + 8\mathbf{j})$N $G = (-2\mathbf{i} - 2\mathbf{j})$N

Since the forces are in Cartesian components, R is found by adding the forces.

Therefore $R = (2\mathbf{i} + 3\mathbf{j})$N



Using Pythagoras to find the magnitude:

$$|R| = \sqrt{2^2 + 3^2}$$

$$|R| = \sqrt{13}$$

And finally the angle θ:

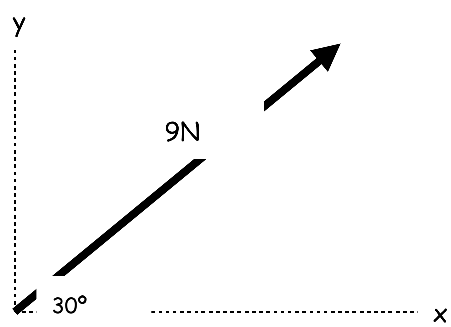
$$\text{Tan}\,\theta = \frac{3}{2}$$

$$\theta = 56.3^\circ$$

Resolving Forces into Components

This concept was introduced in earlier work on vectors as it is far easier to deal with a number of forces if we can split them into their horizontal and vertical components.

The diagram below shows a 9N force acting at an angle of 30° to the horizontal. Find the components of the force in the horizontal and vertical direction.



x component = 9 × Cos 30° = $\frac{9\sqrt{3}N}{2}$

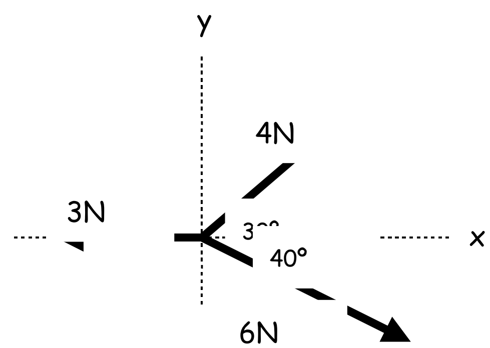
y component = 9 × Sin 30° = $\frac{9}{2}N$

Resolving Several Forces into Components

All we have to do is find all of the x components and add them together to give the x component of the resultant force. This idea is applied to the y component and from this we can find the direction of the resultant.

Example 3

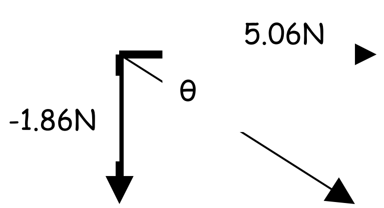
The diagram below shows a number of forces. Find the resultant of the forces and its direction.



The best way to attack this problem is to use a table:

Force	x component	y component
3N	- 3	0
4N	4 × Cos 30°	4 × Sin 30°
6N	6 × Cos 40°	-6 × Sin 40°
Total	5.06	-1.86

The resultant can be represented pictorially as:



The resultant has a magnitude of:

$$= \sqrt{5.06^2 + 1.86^2}$$

$$= 5.39N$$

In the direction of:

$$Tan^{-1}\left(\frac{-1.86}{5.06}\right) = -20.2^\circ$$

Equilibrium of Coplanar Forces

If a system is being acted upon by a series of forces that all lie in the same plane then it will be in equilibrium if their resultant vector is zero.

Example 4

A particle is in equilibrium under the forces (8**i** + 10**j**)N, (-6**i** -5**j**)N and (a**i** + b**j**)N. Find the values of a and b.

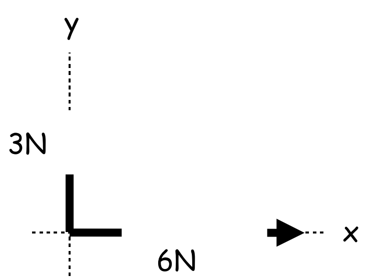
Equating coefficients gives:

$8 + a - 6 = 0$	$10 - 5 + b = 0$
$a = -2$	$b = -5$

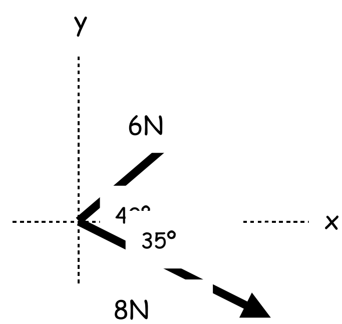
Questions A

1 In each of the following diagrams find the magnitude of the resultant and the angle it makes with the x axis.

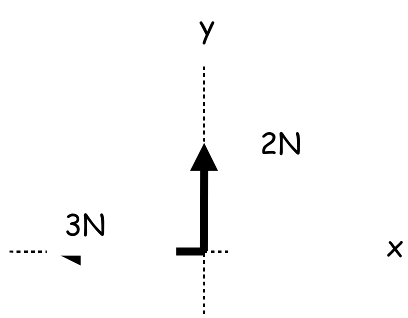
a)



b)



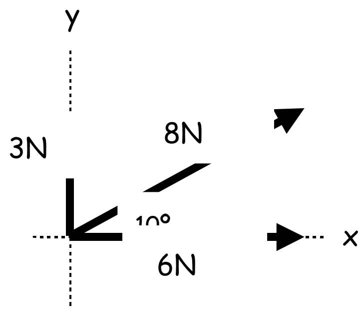
c)

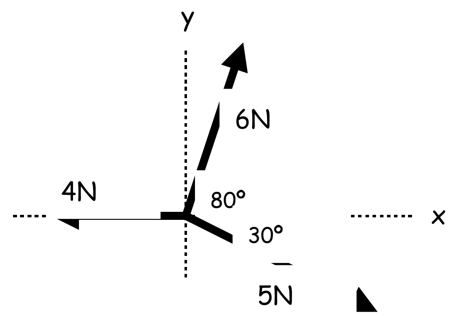


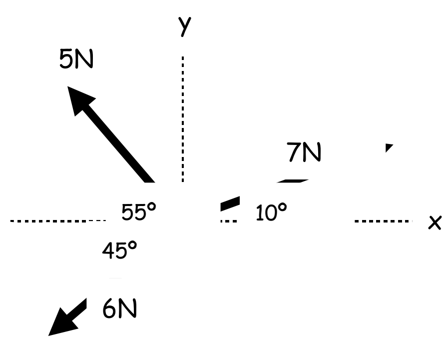
2 Find the angle between a force of 6N and a force of 5N given that their resultant has magnitude 9N.

3 The angle between a force of QN and a force of 3N is 150°. If the resultant of the two forces has magnitude 8N find the value of Q.

4 Each of the following diagrams shows a number of forces. Find the magnitude of their resultant and the angle it makes with the x axis.



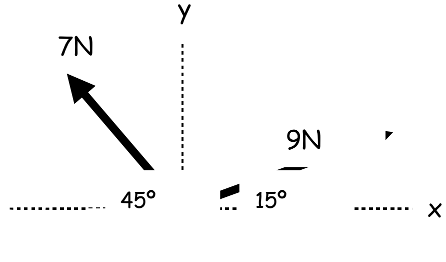




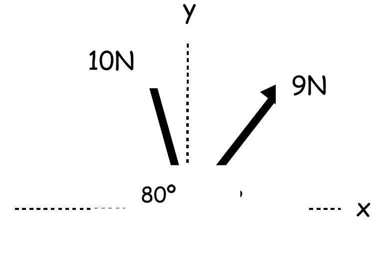
5 For each of the sets of axis below find the sum of the components in the direction of:

a) the x axis b) the y axis

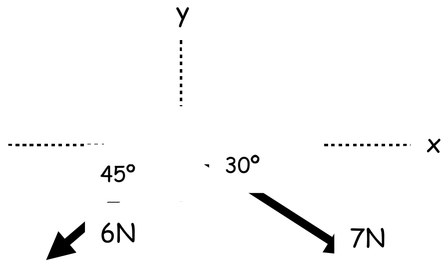
(i)



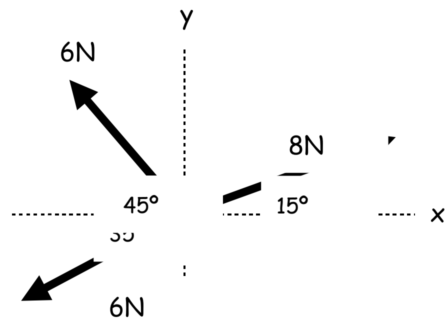
(ii)



(iii)



(iv)



6 Find the magnitude of the force (-3**i** + 8**j**)N and the angle it makes with the direction of **i**.

7 Find the resultant of the following forces:
(2**i** + 4**j**)N, (3**i** - 5**j**)N, (6**i** + 2**j**)N and (-5**i** + 8**j**)N

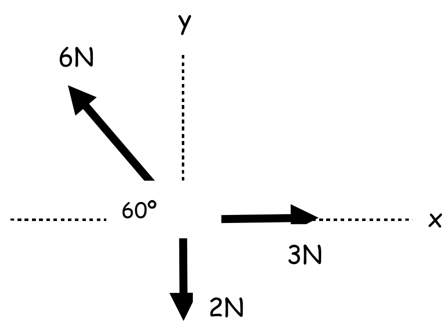
8 Find the magnitude of the resultant in question 7 and find the angle it makes with the direction of **i**.

9 The resultant of the forces (5**i** - 2**j**)N, (7**i** + 4**j**)N, (a**i** + b**j**)N and (-3**i** + 2**j**)N is a force (5**i** + 5**j**)N. Find a and b.

10 Find the resultant of forces 5N, 7N, 8N and 5N in the directions, north, north east, south west and south east respectively, giving your answer in the form a**i** + b**j**

11 Two forces, P and Q, are such that the sum of their magnitudes is 45N. The resultant of P and Q is perpendicular to P and has a magnitude of 15N. Calculate:
a) the magnitude of P and Q;
b) the angle between P and Q.

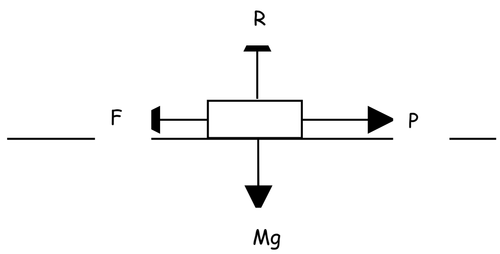
12 Forces of magnitude 3N, 6N and 2N act at a point as shown in the diagram below. Show that the component of the resultant force in the **i** direction is zero. Calculate the magnitude of the resultant force and state its direction.



Friction

Rough and Smooth surfaces

A block of mass M Kg on a horizontal table is acted upon by a force P Newtons. From Newton's Third Law, it is known that equal and opposite forces act on the block and on the plane at right angles to the surfaces in contact.



The force F acts to oppose the motion and this is called the frictional force. If the surface were to be perfectly smooth then the block would accelerate across the surface. In general the surface is unlikely to be smooth and the block would move if the force P was greater than the frictional force.

Limiting Equilibrium

The frictional force in the above situation is not constant, but increases as the force P increases until if reaches a value F_{max}. The block is then on the point of moving and the system is said to be in a state of **limiting equilibrium**.

Coefficient of Friction (μ)

Friction is proportional to the normal reaction and in limiting equilibrium it is given by:

$$F_{max} = \mu R$$

Where μ is the coefficient of friction for the two contact surfaces.

Friction can also be expressed as:

$$F \leq \mu R$$

When friction is less than μR motion will not take place.

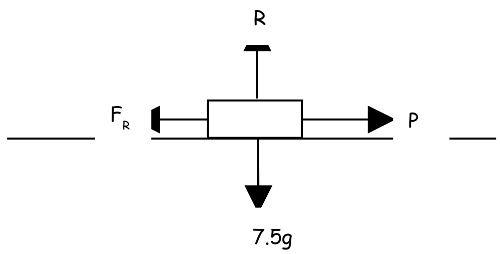
Consider the following points:

1. At the point where friction can't increase any further, motion is about to take place.
2. Note that friction is only dependent upon the nature of the surfaces in contact and not upon the contact area.
3. For perfectly smooth surfaces $\mu = 0$.
4. Friction will never be larger than that necessary to prevent motion.
5. It can be assumed that friction will have a maximum value μR when motion occurs.
6. Friction always acts to oppose the motion of an object and great care must be taken with objects on slopes as friction could be acting either up or down the plane (see examples 7 and 8).

Example 5

A block of mass 7.5Kg rests on a rough horizontal plane, the coefficient of friction between the block and the plane is 0.55. Calculate the frictional force acting on the block when a horizontal force P is applied to the block and the magnitude of any acceleration that may occur.

There is no motion perpendicular to the plane.



a) 15N b)65N

a) Resolving perpendicular to the plane gives:

$$R = 7.5g$$

Friction will act to oppose motion and at its maximum value $F_{max} = \mu R$, therefore:

$$\mu R = 0.55 \times 7.5g$$

$$\mu R = 40.4N$$

In this example it is possible for friction to increase until it reaches a value of 40.4N. There is only a pushing force of 15N therefore friction will prevent motion.

b) Seeing as the pushing force is greater than 40.4N the block will accelerate across the surface. Using the value of μR from above (since we have reached the value of $\mathbf{F}_{max}$) and by setting up an equation of motion for the block we get:

$$F = ma$$

$$65 - 40.4 = 7.5 \times a$$

$$a = 3.28\text{ms}^{-2}$$

Note the difference between F and $\mathbf{F}_{max}$: The F is the sum total of all the forces acting in the horizontal direction.

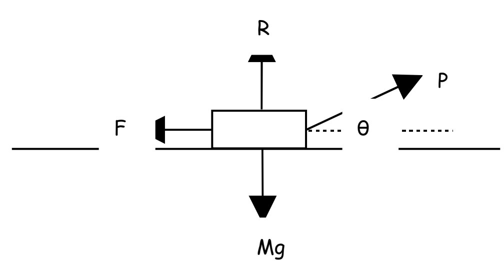
Non Horizontal Forces

When the force P acting on the block of mass M is inclined at an angle to the horizontal, two effects must be considered.

(1) The vertical component of P alters the size of the normal reaction R. One needs to consider the direction of the applied force and its effect on the value of R.

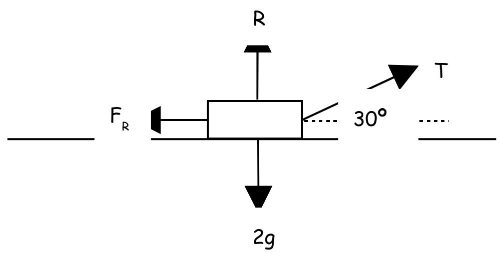
(2) Only the horizontal component of the force P will bring about motion in the block.

It is worth noting at this point that since R has altered there will also be a change in the frictional force (F = μR).



Example 6

A box of mass 2Kg lies on a rough horizontal floor with the coefficient of friction between the floor and the box being 0.5. A light string is attached to the box in order to pull the box across the floor. If the tension in the string is TN, find the value that T must exceed for motion to occur if the string is 30° above the horizontal.



If motion is to take place then $F_R = \mu R$.

Resolving forces perpendicular to the plane gives:

$$2g = R + T \times \text{Sin } 30°$$

$$R = 19.6 - 0.5T$$

Using $F_R = \mu R$

$$F_R = 9.8 - 0.25T \ (1)$$

Resolving parallel to the plane gives:

$$F_R = T \times \text{Cos } 30° \ (2)$$

Equating (1) and (2) gives:

$$9.8 - 0.25T = T\frac{\sqrt{3}}{2}$$

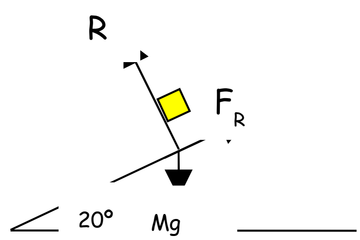
$$T = 8.78N$$

Objects on Inclined Planes

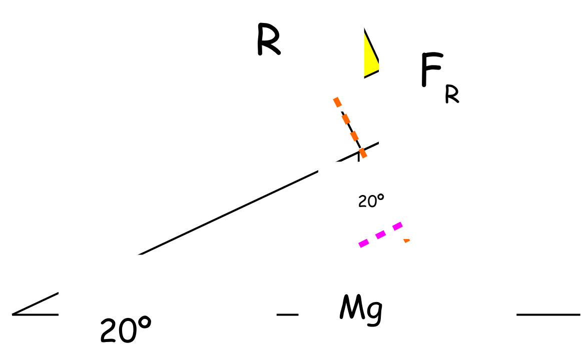
When objects are on inclined planes it is easier to resolve the forces parallel to the plane and perpendicular to the plane. This concept is best shown through an example.

Example 7

A particle of mass MKg rests in equilibrium on a rough plane inclined at an angle 20° to the horizontal. Find the normal reaction R and the frictional force in terms of M and g.



The normal reaction by definition only reacts to the component of the weight force that acts perpendicular to the plane (the orange line in the diagram below). Equally the frictional force is only acting against the component of the weight that is acting parallel to the plane (the pink line).



This leads to the following two statements that will be used over and over again in M1.

$$R = Mg \cos 20^\circ$$

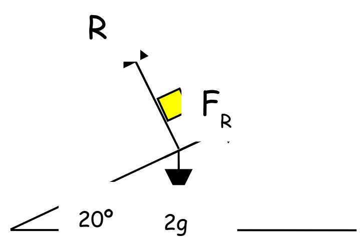
$$\mathbf{F_R} = Mg \sin 20^\circ$$

This is only valid if there are no other external forces.

<u>Varying Values of Friction</u>

In the introduction of the coefficient of friction we suggested that friction can vary. Now that we know how to resolve forces parallel and perpendicular to the plane we can use this new skill to explain the point.

Consider the case below with a 2kg mass on a rough surface inclined at an angle of 20°, where the coefficient of friction between the object and the surface is 0.4.



Resolving forces perpendicular to the plane gives:

$$R = 2g \cos 20^\circ = 18.4\text{N}$$

Therefore:

$$\mu R = 7.37\text{N}$$

Resolving parallel to the plane gives:

$$F_R = 2g \text{ Sin } 20^\circ = 6.70\text{N}$$

The condition $F_R \leq \mu R$ is upheld and as a result there is no motion down the plane.

If the slope was raised to 30°, resolving forces perpendicular to the plane gives:

$$R = 2g \text{ Cos } 30^\circ = 17.0\text{N}$$

Therefore:

$$\mu R = 6.79\text{N}$$

Resolving parallel to the plane gives:

$$F_R = 2g \text{ Sin } 30^\circ = 9.8\text{N}$$

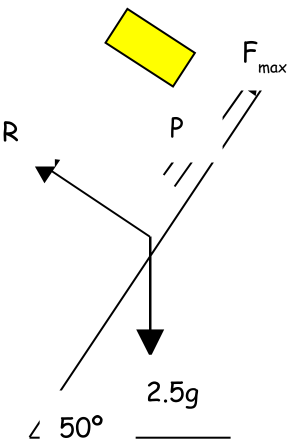
Friction is working against the parallel component of the weight (mg Sin θ). By definition Sin θ increases as the angle increases therefore friction must increase to prevent motion, but it can only increase to the point where $F = \mu R$. In the second part of the example the condition $F \leq \mu R$ is no longer upheld and therefore the object would slide down the slope. The object in the example above would be in a state of limiting equilibrium for an angle between 20° and 30° (calculate the exact value).

Other external forces on inclined planes

Example 8

A mass of 2.5Kg rests on an inclined plane at 50° to the horizontal, and the coefficient of friction between the mass and the plane is 0.3. Find the force P, acting parallel to the plane, which must be applied to the mass in order to just prevent motion down the plane.

Seeing as the mass is about to slide down the plane, friction must act up the plane.



Resolving the forces parallel to the plane gives:

$$F_{max} + P = 2.5g \text{ Sin } 50°$$

Resolving perpendicular to the plane:

$$R = 2.5g \text{ Cos } 50°$$

Using $F_{max} = \mu R$:

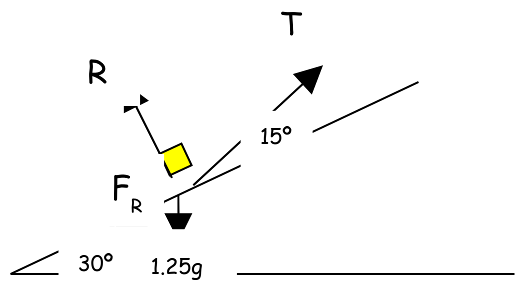
$$0.3 \times (2.5g \text{ Cos } 50°) + P = 2.5g \text{ Sin } 50°$$

$$P = 14.0\text{N}$$

The next problem introduces the idea of forces acting on a particle on an inclined plane where the force acts at an angle to the plane.

Example 9

A building block of mass 1.25kg is placed on an incline plane at an angle of 30° to the horizontal. The coefficient of friction between the box and the plane is 0.2. The box is kept in equilibrium by a light inextensible string which lies in a vertical plane. The string makes an angle of 15° with the plane. The box is in limiting equilibrium and is about to move up the plane. The tension in the string is T Newtons. Modelling the box as a particle, find the value of T.



Note that the frictional force is acting down the slope as the box is at the point of moving up the plane.
Once again we need to resolve the forces into their components, but this time we must resolve them parallel and perpendicular to the plane.

Resolving parallel to the plane:
(don't forget to resolve the tension force)

$$1.25g \times \text{Sin } 30 + F_R = T \times \text{Cos } 15$$

$$6.125 + F_R = 0.966T$$

$$F_R = 0.966T - 6.125 \qquad \textbf{(1)}$$

Resolving perpendicular to the plane:

$$1.25g \times \cos 30 = R + T \times \sin 15$$

$$10.61 = R + 0.259T$$

$$R = 10.61 - 0.259T$$

Using $F = \mu R$

$$0.966T - 6.125 = 0.2 \times (10.61 - 0.259T)$$

$$1.0178T = 8.247$$

$$T = 8.10N$$

Questions B

1 A block of mass 3Kg is initially at rest on a rough horizontal plane. The coefficient of friction between the block and the plane is 0.6. Calculate the pushing force required to cause it to accelerate along the surface at:
a) $8ms^{-2}$ b) $0.1ms^{-2}$

2 A block of mass 12kg rests on a rough horizontal surface. The coefficient of friction between the block and the plane is 0.4. Calculate the frictional force when a horizontal force of 65N acts on the block. If the block moves, find the acceleration.

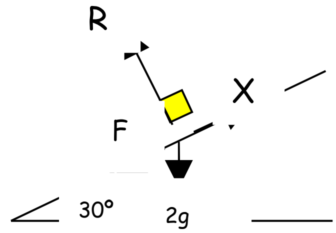
3 A block of mass 7kg rests on a rough horizontal surface. The coefficient of friction between the block and the plane is 0.15. A light string is attached to the box in order to pull the box across the floor. If the string makes an angle of 45° with the horizontal find the value that the tension in the string must exceed in order to bring about motion.

4 Referring to question 3 what value of tension would be required to give the block an acceleration of $3ms^{-2}$.

5 A block of mass 4.5kg rests on a rough horizontal surface. The coefficient of friction between the block and the plane is 0.25. State whether or not the block will slide when the surface is moved horizontally with an acceleration of of:
a) $0.9ms^{-2}$. b) $2ms^{-2}$. c) $3.5ms^{-2}$.

6 For each of the following diagrams below the value of μ = 0.3. In each case calculate the magnitude of the force X if the body is just on the point of moving.

a)



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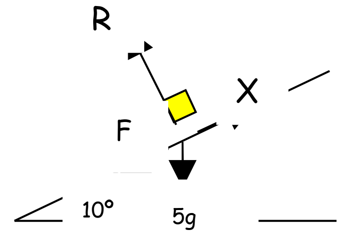
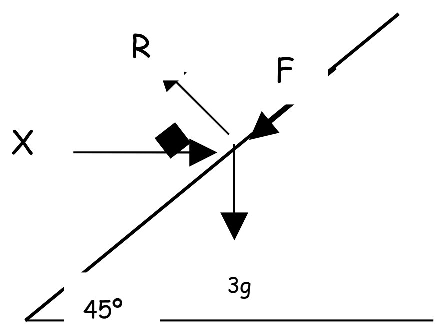
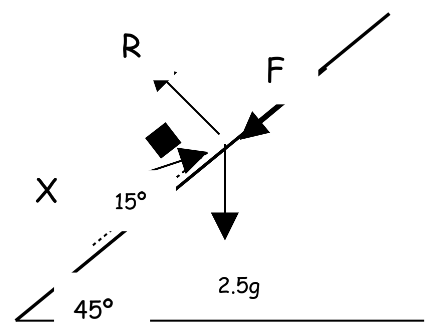


c)



d)



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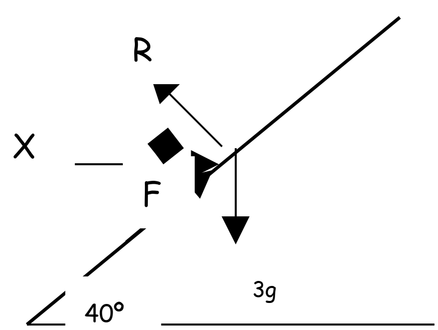


7 A body of mass 6.5kg is placed on an incline plane at an angle of 35° to the horizontal. The coefficient of friction between the box and the plane is 0.2. A horizontal force of 25N is applied to the body. Find the frictional force and the value of the acceleration. State the direction of motion.

8 A body of mass 4kg is released from rest on a rough surface that it inclined at an angle of 40° to the horizontal. If

after 2 seconds the body has a velocity of $4.9ms^{-1}$ down the surface, find the coefficient of friction between the body and the surface.

9 A horizontal force of 0.9N is just sufficient to prevent a brick of mass 750g from sliding down a rough plane inclined at an angle of 20°. Calculate the coefficient of friction between the plane and the brick.

10 A box of mass 5kg is placed on an incline plane at an angle of 35° to the horizontal. The coefficient of friction between the box and the plane is 0.45. Find the horizontal force that must be applied so that:

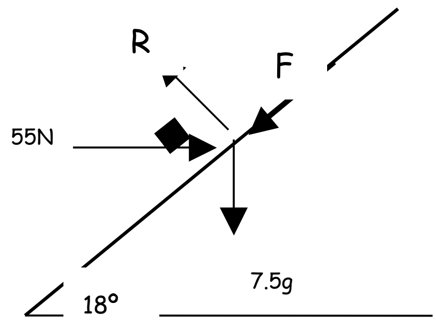
a) the box is just prevented from sliding down the plane

b) the box is just at the point of moving up the plane

c) the box moves up the plane with an acceleration of $1.25ms^{-2}$.

Solutions to Exam type questions

1



A box of mass 7.5kg is held in limiting equilibrium on a rough plane by the action of a horizontal force of magnitude 55N acting in a vertical plane through a line of greatest slope. The plane is inclined at an angle of 18° to the horizontal, as shown in the diagram above. By modeling the box as a particle and noting that it is on the point of moving up the slope:

a) find the normal reaction R.

b) find the coefficient of friction between the box and the plane.

a) Resolving perpendicular to the plane gives:

$$R = 7.5g \times \text{Cos } 18 + 55 \times \text{Sin } 18$$

$$R = 90.5\text{N}$$

(b) Resolving parallel to the plane gives:

$$F + 7.5g \times \text{Sin } 18 = 55 \times \text{Cos } 15$$
$$F = 30.4\text{N}$$

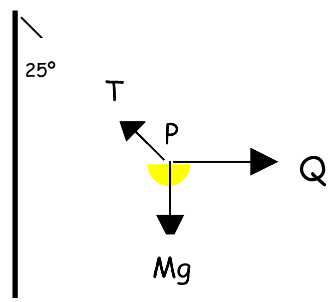
Using $F = \mu R$:

$$30.4 = \mu \times 90.5$$
$$\mu = 0.336$$

2 A swing ball is attached to one end of a light inextensible string; the other end of the string is attached to the top of a fixed vertical pole. A child applies a horizontal force, Q, of magnitude 45N to the particle P. P is in equilibrium under gravity with the string making an angle 25° with the vertical pole. By modeling the ball as a particle find:

a) the tension, T, in the string.

b) the weight of the ball.



a) All problems of this type can be solved by resolving forces into their components.

Resolving horizontally:

$$Q = T\cos 65°$$

$$T = \frac{45}{\cos 65°}$$

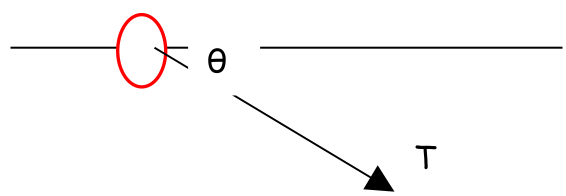
$$T = 106\text{N}$$

b) Resolving vertically:

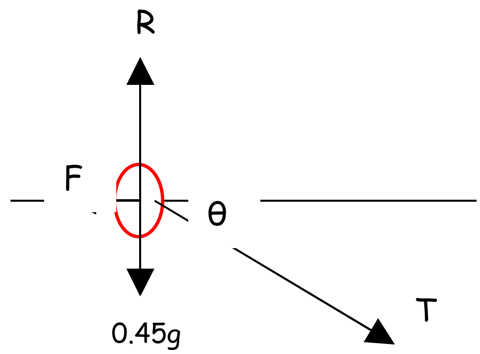
$$Mg = T\sin 65°$$

$$Mg = 96.5\text{N}$$

3 A ring of mass 0.45kg is threaded onto a fixed, rough horizontal pole. A light inextensible string is attached to the ring. The string and the pole lie in the same vertical plane. The ring is pulled downwards by the string which makes an angle θ with the horizontal, where tan θ = $\frac{3}{4}$. The tension in the string is 2.2N. Find the coefficient of friction between the ring and the pole.



Adding the weight, the reaction and frictional forces to the diagram gives:



Resolving vertically gives:

$$0.45g + T \sin \theta = R$$

Given that Tan θ = $\frac{3}{4}$ hence Sin θ = $\frac{3}{5}$

Therefore: $R = 0.45g + 2.2 \times 0.6$

$$R = 5.73N$$

Resolving horizontally gives:

$$T \cos \theta = F$$

Given that Tan $\theta = \frac{3}{4}$ hence Cos $\theta = \frac{4}{5}$

$$F = 2.2 \times 0.8$$

$$F = 1.76$$

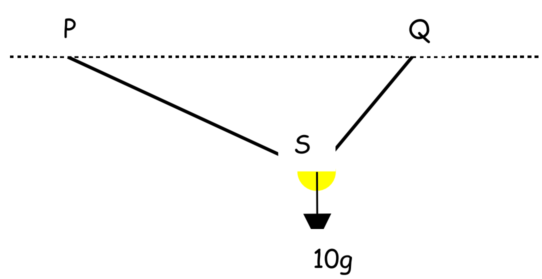
Using $F = \mu R$

$$1.76 = \mu \times 5.73$$

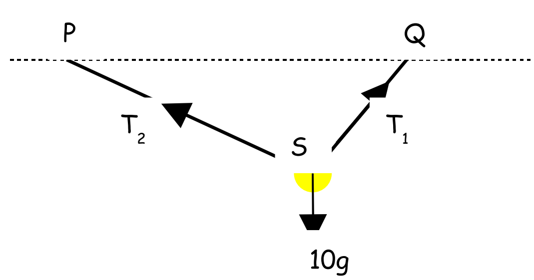
$$\mu = 0.307$$

4 A particle of weight 98 is attached at S to the ends of two light inextensible strings PS and QS. The other ends of the string are attached to two fixed points P and Q on a horizontal ceiling. The particle hangs in equilibrium with PS and QS inclined to the horizontal at angles of 25 and 60 degrees respectively.

Calculate the tension in both strings.



Adding the two tension forces to the diagram gives:



Seeing as the object is in equilibrium then the forces must resolve to zero

Resolving horizontally gives:

$$T_2 \text{ Cos } 25° = T_1 \text{ Cos } 60°$$

Therefore: $T_2 = 0.552T_1$ (1)

Resolving horizontally gives:

$$T_2 \text{ Sin } 25° + T_1 \text{ Sin } 60° = 10g$$

Using (1) from above:

$$\text{Sin } 25° \times 0.552T_1 + T_1 \text{ Sin } 60° = 10g$$

$$T_1 = 89.2\text{N}$$

Therefore: $T_2 = 49.2\text{N}$