SIMPLE HARMONIC MOTION

1. a) $t = 0 \Rightarrow x = 4 - 3\sin 0$

Distance is 4m — B1

$v = \frac{dx}{dt} = -\frac{3\pi}{2}\cos\frac{\pi t}{2}$ — M1 A1

$\therefore$ At rest if $\cos\frac{\pi t}{2} = 0$ i.e. $\frac{\pi t}{2} = \frac{\pi}{2}, \frac{3\pi}{2}...$ — M1

$\therefore$ t = 1, 3, 5 i.e. t = (2n–1), where n is an integer ≥ 1 — A1

[5]

b) $v = \frac{dx}{dt} = -\frac{3\pi}{2}\cos\frac{\pi t}{2}$

$a = \frac{dv}{dt} = -\frac{3\pi^2}{4}\sin\frac{\pi t}{2}$ — M1 A1

Distance of particle from (4,0) $= 4 - 3\sin\frac{\pi t}{2} - 4$

$= -3\sin\frac{\pi t}{2}$ — B1

So a = $-\frac{\pi^2}{4} \times$ distance — B1

$\Rightarrow$ S.H.M.

amplitude 3 — B1

Period = $\frac{2\pi}{\frac{\pi}{2}}$ = 4 seconds — B1

[6]

SIMPLE HARMONIC MOTION

2. a) Since in equilibrium, no resultant vertical force $\therefore T = 6g$ M1

By Hooke's Law: $T = \frac{10gx}{2}$ M1

$6g = 5gx \Rightarrow x = 1.2$ A1

Stretched length of AB = 3.2 m A1 f.t.

[4]

b) i) Let particle be at a general point, distance y below equilibrium

Then $T = \frac{10g(1.2 + y)}{2}$ M1 A1

Resolving $\downarrow$: $6g - T = 6\ddot{y}$ M1 A1

$\Rightarrow 6g - 6g - 5gy = 6\ddot{y}$

$-\frac{5gy}{6} = \ddot{y}$ B1

This is of form $\ddot{y} = -\omega^2 y$, so it is SHM

Particle performs complete oscillations if string does not go slack – so $a \le 1.2$ m B1

[6]

ii) $\omega = \sqrt{\frac{5g}{6}}$ $a = 1$

$\Rightarrow V\ max = \sqrt{\frac{5g}{6}}$ B1 f.t.

$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{6}{5g}}$ B1 f.t.

[2]

SIMPLE HARMONIC MOTION

3. a)	Hooke's law $T = \frac{10x}{5}$ i.e. $T = 2x$	M1 A1
	$T = 4N$	A1
	$4 = 2x \quad x = 2$	A1 cao
	$\therefore$ length of spring is $5 - 2 = 3$m	B1 f.t.
		[5]
b)	Let the compression of the spring be x	
	Resolving in direction $\overrightarrow{BA}$:	M1
	$1 \times \ddot{x} = \frac{-10x}{5}$	A1
	$\ddot{x} = -2x$	A1
	so SHM, $\omega^2 = 2 \quad a = 2$	B1 (may be implied)
	use $x = a\sin\omega t$: $\Rightarrow x = 2\cos\sqrt{2}t$	M1 A1 A1
	At natural length, $x = 0 \Rightarrow \cos\sqrt{2}t = 0$	M1
	$\sqrt{2}t = \frac{\pi}{2} \Rightarrow t = \frac{\pi}{2\sqrt{2}}$	A1
		[9]

SIMPLE HARMONIC MOTION

4. a) For AB: $T = \frac{\lambda x}{L} \Rightarrow T_A = \frac{\lambda_A L}{2L} = \frac{\lambda_A}{2}$ M1 A1

For BC: x = 2L A1 cao

$\therefore T_C = \frac{\lambda_C 2L}{L} = 2\lambda_C$ A1 cao

but $T_A = T_C$ M1

$\therefore \frac{\lambda_A}{2} = 2\lambda_C$ i.e. $\lambda_A : \lambda_C = 4:1$ A1 f.t.

[6]

b) Consider particle distance y from equilibrium, in direction $\overrightarrow{CA}$

$T_A = \frac{\lambda_A (L - y)}{2L}$ M1 A1

$T_C = \frac{\lambda_C (2L + y)}{L}$ A1

Resolving: $T_A - T_C = \ddot{y}$ M1

$\frac{4\lambda_C L}{2L} - \frac{4\lambda_C y}{2L} - \frac{2L\lambda_C}{L} - \frac{\lambda_C y}{L} = \ddot{y}$ A1

$\frac{-3\lambda_C y}{L} = \ddot{y} \Rightarrow$ of form $\ddot{y} = -\omega^2 y$ B1

$\Rightarrow$ SHM

[6]

c) Specified point is $\frac{L}{4}$ from equilibrium B1

$v^2 = \omega^2(a^2 - x^2)$ M1

$= \frac{3\lambda_C}{L}\left(\frac{L^2}{4} - \frac{L^2}{16}\right)$ A1

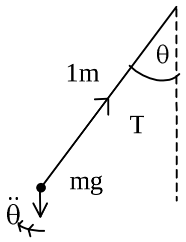
$= \frac{9\lambda_C L}{16}$ A1

$v = \frac{3}{4}\sqrt{L\lambda_C}$ A1

[5]

SIMPLE HARMONIC MOTION

5. a) Consider the particle at angle θ from equilibrium position



Resolving perpendicular to string: $m \times 1 \times \ddot{\theta} = -mg\sin\theta$	M1 A1 A1
$\Rightarrow \ddot{\theta} = -g\sin\theta$	A1
θ small $\Rightarrow \sin\theta \approx \theta$	B1
$\ddot{\theta} = -g\theta \Rightarrow$ SHM	
Period $\dfrac{2\pi}{\omega} = \dfrac{2\pi}{\sqrt{g}}$	B1 f.t.
	[6]

b) Use $\theta = \alpha\cos(\sqrt{g}\,t)$ (since $t = 0 \Rightarrow \theta = \alpha$)	M1
$\theta = \frac{\alpha}{2} \Rightarrow \frac{1}{2} = \cos(\sqrt{g}\,t)$	M1
$\frac{\pi}{3} = (\sqrt{g}\,t) \Rightarrow t = \dfrac{\pi}{3\sqrt{g}}$	A1
$\theta = -\frac{\alpha}{2} \Rightarrow -\frac{1}{2} = \cos(\sqrt{g}\,t) \Rightarrow t = \dfrac{2\pi}{3\sqrt{g}}$	A1
so between these points for $\dfrac{2\pi}{3\sqrt{g}} - \dfrac{\pi}{3\sqrt{g}} = \dfrac{\pi}{3\sqrt{g}}$ seconds	B1
same on reverse trip $\Rightarrow$ altogether $\dfrac{2\pi}{3\sqrt{g}}$ seconds	A1
	[6]

c) $\dot{\theta}_{max} = \omega a = \sqrt{g}\,\alpha$	M1 A1
	[2]

d) Conservation of momentum:	M1
$m\sqrt{g}\,\alpha \times 1 = 3mv \Rightarrow v = \dfrac{\sqrt{g}\,\alpha}{3}$	A1
Use conservation of energy:	M1
Let P.E. = 0 at lowest point	
Initially: K.E. = $\frac{1}{2} \times 3m \times \dfrac{g\alpha^2}{9} = \dfrac{mg\alpha^2}{6}$ P.E. = 0	A1
Finally: K.E. = 0 P.E. = 3mgh	A1
$3\,mgh = \dfrac{mg\alpha^2}{6} \Rightarrow h = \dfrac{\alpha^2}{18}$ m	A1
	[6]

SIMPLE HARMONIC MOTION

6. a) $T = \dfrac{12mg\,x}{2L}$ M1

$6mg = \dfrac{12mg\,x}{2L}$ M1

L = x A1 cao

Length = 3L A1 f.t.

[4]

b) Consider particle distance x below equilibrium:
Resolving downwards:

$6mg - \dfrac{12mg(L+x)}{2L} = 6\text{ ma}$ M1 A1

$6mg - 6mg - \dfrac{6mg\,x}{L} = 6ma$ A1

$\dfrac{-gx}{L} = a$ A1

$\Rightarrow$ SHM initially

not complete oscillations because string will go slack when it is L above equilibrium B1

[5]

c) Still performs SHM until this point, so use $v^2 = \omega^2 (a^2 - x^2)$ M1

$a = 2L \quad \omega^2 = \frac{g}{L} \quad x = -L$ B1

$v^2 = \frac{g}{L}(4L^2 - L^2)$

$v^2 = 3gL$

$v = \sqrt{3gL}$ A1

Use x = a cos ωt since initially x = 2L M1

$x = 2L\cos\left(\sqrt{\dfrac{g}{L}}\,t\right)$

$-L = 2L\cos\left(\sqrt{\dfrac{g}{L}}\,t\right)$ A1

$-\frac{1}{2} = \cos\left(\sqrt{\dfrac{g}{L}}\,t\right)$

$\dfrac{2\pi}{3} = \sqrt{\dfrac{g}{L}}\,t$ A1

$t = \dfrac{2\pi}{3}\sqrt{\dfrac{L}{g}}$ A1

[7]

SIMPLE HARMONIC MOTION

QUESTION 6 CONTINUED

d) Conservation of energy after string goes slack: M1
P.E. = 0 at point where string goes slack
$\Rightarrow$ mgh = $\frac{1}{2}$ m 3gL A1

h = $\frac{3L}{g}$ $\Rightarrow$ $\frac{L}{2}$ below fixed point A1

[3]

e) Time between string going slack and particle reaching highest point:
v = u + at M1
0 = $\sqrt{3gL} - gt$

t = $\frac{\sqrt{3gL}}{g} = \sqrt{\frac{3L}{g}}$ A1

Total time = $2\left(\frac{2\pi}{3}\sqrt{\frac{L}{g}} + \sqrt{\frac{3L}{g}}\right)$ M1 M1 A1

$\left(= \frac{2}{3}\sqrt{\frac{L}{g}}\left(2\pi + 3\sqrt{3}\right)\right)$

[5]

SIMPLE HARMONIC MOTION

7. a) Resolving : $mg = \frac{kmgx}{L}$ M1 A1

$\frac{L}{k} = x$ A1

[3]

b) i) Consider particle distance y below equilibrium:

Resolving $\downarrow$: $mg - \frac{kmg\left(\frac{L}{k} + y\right)}{L} = m\ddot{y}$ M1 A1

$-\frac{kmg}{L}y = m\ddot{y}$

$-\frac{kg}{L}y = \ddot{y}$ A1

Period = $\frac{2\pi}{\sqrt{\frac{kg}{L}}} = 2\pi\sqrt{\frac{L}{kg}}$ M1 A1

$2\pi\sqrt{\frac{L}{kg}} = \frac{10\pi}{7}$

$\sqrt{\frac{L}{kg}} = \frac{5}{7} \Rightarrow \frac{L}{kg} = \frac{25}{49}$ A1

$k = \frac{49L}{25g} = \frac{L}{5}$ A1

[7]

ii) $v_{max} = \omega a$ M1

$= \sqrt{\frac{kg}{L}}\,\frac{L}{2}$ A1

$= \sqrt{\frac{g}{5}}\,\frac{L}{2}$

$= \frac{7L}{10}$ A1 f.t.

Max acceleration = $\omega^2 a$ M1

$= \frac{kg}{L}\,\frac{L}{2}$

$= \frac{gL}{10}$ A1 f.t.

[5]

SIMPLE HARMONIC MOTION

8.	a) Force = -kx	B1
	When x = 2. force = 36	M1
	$\Rightarrow 36 = 2k \Rightarrow k = 18$	A1
	Using F = ma: $2\ddot{x} = -18x$	M1
	Giving $\ddot{x} = -9x$, or $\ddot{x} + 9x = 0$	A1
		[5]
	b) It is SHM	M1
	$v_{max} = \omega a$, where a is the amplitude	M1
	$\omega = 3$	B1
	$\Rightarrow 12 = 3a \Rightarrow 4 = a$	A1
		[4]
	c) $x = 4\cos(3t + \alpha)$.	M1
	When t = 0, x = 2, so:	
	$2 = 4\cos\alpha$	
	$\frac{1}{2} = \cos\alpha$	A1
	$\alpha = \pm\frac{\pi}{3}$	A1
	Since moving <u>towards</u> origin, require $\dot{x}$ negative	M1
	$\dot{x} = -12\sin(3t + \alpha)$	M1
	$t = 0,\ \dot{x} = -12\sin\alpha$	
	So $\sin\alpha$ is positive	
	$\Rightarrow \alpha = \frac{\pi}{3}$	A1
	$x = 4\cos(3t + \frac{\pi}{3})$	
	$x = 0 \Rightarrow \cos(3t + \frac{\pi}{3}) = 0$	M1
	$3t + \frac{\pi}{3} = \frac{\pi}{2}, \frac{3\pi}{2}$	A1 (either)
	$\Rightarrow 2^{nd}$ time: $3t + \frac{\pi}{3} = \frac{3\pi}{2}$	A1
	$3t = \frac{7\pi}{6}$	
	$t = \frac{7\pi}{18}$ s	A1
		[10]

SIMPLE HARMONIC MOTION

9. a)	$T = \dfrac{2\pi}{\omega}$	M1
	$\omega = \dfrac{2\pi}{\frac{\pi}{8}}$	
	$= 16$	A1
	$v_{max} = \omega a$	M1
	$= 16 \times 0.6$	
	$= 9.6 \text{ ms}^{-1}$	A1
		[4]
b)	$x = 0.6\sin 16t$	M1 A1
	At B: $-0.4 = 0.6\sin 16t$	M1
	$\sin 16t = -\frac{2}{3}$	A1
	$16t = 3.871, 5.553$	A1
	Will be 2nd time, since it must be moving towards 0	M1(may be implied)
	$T = 0.347$ s. (3SF)	A1 cao
		[7]

SIMPLE HARMONIC MOTION

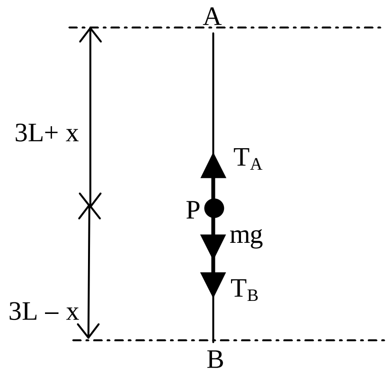
10. a)	Use $v^2 = \omega^2(a^2 - x^2)$	M1
	$v_{max} = \omega a$	M1
	$\Rightarrow 108 = \omega^2(a^2 - 9)$	A1
	$12 = \omega a$	A1
	$\Rightarrow 108 = \omega^2 a^2 - 9\omega^2$	M1 (solving)
	$108 = 144 - 9\omega^2$	
	$36 = 9\omega^2$	
	$2 = \omega$	A1
	$T = \dfrac{2\pi}{\omega} = \pi$	B1
	$a = 6$	A1
		[8]
b)	$x = a\sin(\omega t + \varepsilon)$	M1
	$= 6\sin(2t + \varepsilon)$	
	$t = \frac{\pi}{3}$, $3 = 6\sin\left(\dfrac{2\pi}{3} + \varepsilon\right)$	
	$\frac{1}{2} = \sin\left(\dfrac{2\pi}{3} + \varepsilon\right)$	A1
	So $\frac{2\pi}{3} + \varepsilon = \frac{\pi}{6}$ or $\frac{5\pi}{6}$	A1 (either)
	Must have negative velocity	M1
	$\dot{x} = 12\cos(2t + \varepsilon)$	A1
	so must have $\frac{2\pi}{3} + \varepsilon = \frac{5\pi}{6}$, to make this negative	A1
	$\varepsilon = \frac{5\pi}{6} - \frac{2\pi}{3} = \frac{\pi}{6}$	A1
	$x = 6\sin(2t + \frac{\pi}{6})$.	
	Passes through origin $\Rightarrow x = 0$	M1
	$\sin(2t + \frac{\pi}{6}) = 0$	
	$2t + \frac{\pi}{6} = \pi, 2\pi\ldots$	A1 (either)
	Since speed positive, $2t + \frac{\pi}{6} = 2\pi$	M1 A1
	$t = \frac{11\pi}{12}$	A1
		[12]

SIMPLE HARMONIC MOTION

11.a)	$\dot{\mathbf{r}} = 2(-2\sin 2t\,\mathbf{i} + B\cos Bt\,\mathbf{j})$	M1 A1
	$\ddot{\mathbf{r}} = 2(-4\cos 2t\,\mathbf{i} - B^2\sin Bt\,\mathbf{j})$	A1
	$2(-4\cos 2t\,\mathbf{i} - B^2\sin Bt\,\mathbf{j}) = -k^2 2(\cos 2t\,\mathbf{i} + \sin Bt\,\mathbf{j})$	M1
	Equating **i** coefficients $\Rightarrow k^2 = 4$	M1
	Equating **j** coefficients $\Rightarrow B^2 = 4 \Rightarrow B = 2$	A1
		[6]
b)	Craig's position is given by the coefficient of **i** in Dan's position vector	B1
	This therefore satisfies $\ddot{x} = -k^2x$, from part a)	M1
	So period $= \dfrac{2\pi}{k} = \dfrac{2\pi}{2} = \pi$	M1 A1 f.t.
	Since $x = 2\cos 2t$, amplitude = 2	B1
		(-1 in total if **j** coeff used instead)
		[5]
c)	$\dot{x} = -4\sin 2t$	
	Speed is half its maximum value when $-4\sin 2t = \pm 2$	M1
	Hence $\sin 2t = \pm 0.5$	A1
	First occurs when $t = \dfrac{\pi}{12}$	A1
	Distance between them will be 5 + Dan's y-coordinate	M1 A1
	So distance $= 5 + 2\sin\dfrac{\pi}{6} = 6$ metres	A1
		[6]

SIMPLE HARMONIC MOTION

12.a)



Resolving: $T_A = mg + T_B$ M1

Let extension of AP be L + x. Then extension of BP = L – x

Hooke's Law $\Rightarrow T_A = \dfrac{2mg(L+x)}{L}$ and $T_B = \dfrac{2mg(L-x)}{L}$ M1

Hence $\dfrac{2mg(L+x)}{L} = mg + \dfrac{2mg(L-x)}{L}$ M1

Giving $x = \dfrac{L}{4}$ A1

So distance above B is $\dfrac{11L}{4}$ A1

[5]

b) Consider motion of particle at a general time after it was released:

Let displacement of particle above equilibrium position be x.

Then $T_A = \dfrac{2mg(1.25L-x)}{L}$ and $T_B = \dfrac{2mg(0.75L+x)}{L}$ M1

Using F = ma upwards: M1

$$\frac{2mg(1.25L-x)}{L} - \frac{2mg(0.75L+x)}{L} - mg = m\ddot{x}$$ A1

Expanding and simplifying gives $-\dfrac{4gx}{L} = \ddot{x} \Rightarrow$ SHM A1

Period is $\dfrac{2\pi}{\sqrt{\frac{4g}{L}}} = \pi\sqrt{\dfrac{L}{g}}$ M1 A1

Amplitude is 2L B1

[7]

c) $AP = 3L \Rightarrow x = \dfrac{L}{4}$

Using $v^2 = \omega^2(a^2 - x^2)$ M1

$$v^2 = \frac{4g}{L}\left[4L^2 - \left(\frac{L}{4}\right)^2\right]$$ A1

$$v = \frac{3}{2}\sqrt{7gL}$$ A1 f.t.

[3]