MOTION UNDER VARIABLE ACCELERATION

1. a) $a = \frac{k}{v} \Rightarrow 4 = \frac{k}{1}$ i.e. $k = 4$	M1 A1
	[2]
b) $a = \frac{4}{v} \Rightarrow v\frac{dv}{dx} = \frac{4}{v}$	B1
$v^2\,dv = 4\,dx$	M1
$\therefore \frac{v^3}{3} = 4x + C$	A1
$x = 0 \quad v = 1 \Rightarrow C = \frac{1}{3}$	
$v^3 = 12x + 1$	A1 f.t
$\therefore$ when $x = 4 \quad v = 3.66\text{ms}^{-1}$ (3 S.F.)	M1 A1 c.a.o.
	[6]
c) $\frac{dv}{dt} = \frac{4}{v}$	B1
$v\,dv = 4\,dt \quad \therefore \quad \frac{v^2}{2} = 4t + C$	M1 A1
$t = 0 \quad v = 1 \Rightarrow C = \frac{1}{2}$	
$v^2 = 8t + 1$	A1 c.a.o.
$\frac{dx}{dt} = \sqrt{8t+1}$	M1
$x = \frac{1}{12}(8t+1)^{\frac{3}{2}} + C$	A1
$t = 0 \quad x = 0 \Rightarrow C = -\frac{1}{12}$	A1 f.t
$\therefore \quad x = \frac{(8t+1)^{\frac{3}{2}} - 1}{12}$	A1 ca.o.
	[8]

MOTION UNDER VARIABLE ACCELERATION

2. a)	When moving upwards the only force is gravity $\Rightarrow$ use equations of motion.	B1
	$u = 7 \quad v = 0 \quad a = -9.8$	B1
	$v^2 = u^2 + 2as \quad \Rightarrow \quad s = 2.5$	M1 A1
	$\therefore$ Height above ground $= 1 + 2.5 = 3.5$	A1
		[5]
b)	For journey down taking downwards as the positive direction	
	resistance $= 0.25v^2$	B1
	$a = 0.25g - 0.25v^2$	M1 A1
	$a = v\dfrac{dv}{dx}$	M1
	$\displaystyle\int \frac{v}{g - v^2}\,dv = \int 0.25\,dx$	M1
	$-\dfrac{1}{2}\ln\left\lvert g - v^2\right\rvert = 0.25x + C$	A1
	$v = 0 \quad x = 0$ (“measuring” from greatest height)	B1
	$\Rightarrow \quad C = -\dfrac{1}{2}\ln g$	A1 f.t
	$\therefore \ln\left\lvert\dfrac{g}{g - v^2}\right\rvert = 0.5x$	
	$\dfrac{g}{g - v^2} = e^{0.5x}$	M1
	$\dfrac{g - v^2}{g} = e^{-0.5x}$	
	$v^2 = g(1 - e^{-0.5x})$	A1 f.t
	$x = 3.5 \quad \Rightarrow \quad v = 2.85$ (3 s.f.)	A1 ca.o.
		[11]

MOTION UNDER VARIABLE ACCELERATION

3. a) Taking downwards as positive direction

$a = 2g - kv$ M1 A1

At limiting speed $a = 0$ B1

$\therefore\ 2g - gk = 0 \Rightarrow k = 2$ A1

[4]

b) $\therefore\ a = 2(g - v)$

i.e. $\frac{dv}{dt} = 2(g - v)$ M1

$\int \frac{dv}{g - v}\,dv = \int 2\ dt$ M1

$-\ln|g - v| = 2t + C$ A1

$t = 0\ \ v = 0 \Rightarrow C = -\ln g$

$\ln\left|\frac{g - v}{g}\right| = -2t$ M1

$\frac{g - v}{g} = e^{-2t} \Rightarrow v = g(1 - e^{-2t})$ A1

$\therefore\ t = 3 \Rightarrow v = g(1 - e^{-6})$ A1 f.t.

$v = \frac{dx}{dt} = g(1 - e^{-2t})$ M1

$x = g\left(t + \frac{e^{-2t}}{2}\right) + C$ A1

$t = 0\ \ x = 0 \Rightarrow C = \frac{-g}{2}$ $\therefore\ x = \frac{g}{2}\left(2t + e^{-2t} - 1\right)$ M1 A1

$t = 3\ \ x = \frac{g}{2}(5 + e^{-6})$ A1 f.t.

[11]

MOTION UNDER VARIABLE ACCELERATION

4. a)	$x = -2e^{-t} + 3t + C$	M1
	$t = 0, \quad x = -2 \Rightarrow -2 = -2 + C$, so $C = 0$	M1
	$x = -2e^{-t} + 3t$	A1
	$a = -2e^{-t}$	B1
		[4]
b)	$t = \ln 5, \quad v = 2e^{-\ln 5} + 3 = 3.4$	M1 A1
	$x = -2e^{-\ln 5} + 3\ln 5 = 3\ln 5 - \frac{2}{5}$	A1
	So: between $t = 0$ and $t = \ln 5$, distance moved $= 3\ln 5 - \frac{2}{5} - (-2)$	M1
	$= 3\ln 5 + 1\frac{3}{5}$	A1
	Between $t = \ln 5$ and $t = 2$: $a = -6$ $\quad u = 3.4$ $\quad t = 2 - \ln 5$	B2
	$s = ut + \frac{1}{2}at^2$	M1
	$= 3.4(2 - \ln 5) + \frac{1}{2}(-6)(2 - \ln 5)^2$	A1
	$= 0.8703$	A1
	So total $= 3\ln 5 + 1.6 + 0.8703 = 7.30$ m (3 S.F.)	A1
		[11]

MOTION UNDER VARIABLE ACCELERATION

5. Newton II $\Rightarrow \dfrac{kmvu}{(x+k)^2} = -ma \qquad \therefore\ a = \dfrac{-kvu}{(x+k)^2}$ M1 A1

$a = v\dfrac{dv}{dx}$ B1

$v\dfrac{dv}{dx} = \dfrac{-kvu}{(x+k)^2}$ i.e. $\dfrac{dv}{dx} = \dfrac{-ku}{(x+k)^2}$ A1 f.t.

$\therefore\ v = \dfrac{ku}{(x+k)} + C$

$v = u$ when $x = 0 \Rightarrow C = 0$ M1

$\therefore\ v = \dfrac{ku}{(x+k)}$ A1

$\therefore$ when speed is halved $\dfrac{u}{2} = \dfrac{ku}{(x+k)}$ M1

i.e. $x = k \Rightarrow$ distance is k m from O M1 A1

$v = \dfrac{dx}{dt} \quad \therefore \quad \dfrac{dx}{dt} = \dfrac{ku}{(x+k)}$ M1

$\int (x+k)\,dx = \int ku\,dt$ M1

$\dfrac{x^2}{2} + kx = kut + D$ A1 f.t.

$t = 0,\ x = 0 \Rightarrow D = 0$ A1 f.t.

$kut = \dfrac{x^2 + 2kx}{2}$

$\therefore\ x = k \Rightarrow t = \dfrac{3k}{2u}$ M1 A1 f.t.

[15]

MOTION UNDER VARIABLE ACCELERATION

6. a) $a = -2v$ — B1

$\therefore \dfrac{dv}{dt} = -2v \qquad \int \dfrac{1}{v}\,dv = \int -2\,dt$ — M1 M1

$\ln|v| = -2t + C$ — A1

$v = 4 \quad t = 0 \quad \Rightarrow \quad C = \ln 4$ — M1

$\therefore \ln \dfrac{v}{4} = -2t \qquad$ i.e. $\quad v = 4e^{-2t}$ — M1 A1

[7]

b) Since $v = 4e^{-2t}$ and $e^{-2t} > 0$ particle will never come to rest. — B1

Maximum value of e^{-2t} is when $t = 0$ $\therefore$ speed will decrease tending to zero. — B1

$a = -2v$ — M1

$\Rightarrow v\dfrac{dv}{dx} = -2v \qquad \dfrac{dv}{dx} = -2$ — A1

$v = -2x + C \qquad v = 4 \quad x = 2 \quad \Rightarrow \quad C = 8$

$\therefore v = -2x + 8$

$x = 4 - \frac{1}{2}v$ — M1 A1

Since $v > 0$, $x < 4$ — M1 A1

[8]

MOTION UNDER VARIABLE ACCELERATION

7. a) Since unit mass $a = g - kv^2$ — M1

For terminal speed $a = 0$ — B1

$\therefore \quad g = 4k \qquad$ i.e. $\quad a = g - \dfrac{gv^2}{4}$ — A1

[3]

b) $a = g - \dfrac{gv^2}{4} = \dfrac{g}{4}\left(4 - v^2\right)$

$a = \dfrac{dv}{dt} = \dfrac{g}{4}\left(4 - v^2\right)$ — M1 A1

$\displaystyle\int \frac{dv}{4 - v^2} = \int \frac{g}{4}\,dt$

$\displaystyle\frac{1}{4}\int\left(\frac{1}{2 - v} + \frac{1}{2 + v}\right)dv = \int \frac{g}{4}\,dt$ — M1 A1

$\ln\left|\dfrac{2 + v}{2 - v}\right| = gt + C$ — M1 A1

$v = 0 \quad t = 0 \quad \Rightarrow \quad C = 0$ — A1 f.t.

$\therefore \quad \dfrac{2 + v}{2 - v} = e^{gt}$

$2 + v = 2e^{gt} - ve^{gt}$

$v = \dfrac{2\left(e^{gt} - 1\right)}{e^{gt} + 1}$ — A1 f.t

$t = 1 \Rightarrow v = 2.00\text{ms}^{-1}$ (3 S.F.) — A1 ca.o.

[9]

MOTION UNDER VARIABLE ACCELERATION

8. a)	Resultant force = -F – 2x	M1
	Since moving F = μN	B1
	$\therefore$ resolving vertically and no vertical motion $\Rightarrow$ N = 0.5g	B1
	$\therefore$ F = 2	A1 f.t.
	$\therefore$ Newton II $\Rightarrow$ -2 – 2x = 0.5a	M1
	a = -4(1 + x)	A1
		[6]
b)	Impulse = change in momentum	M1
	$\therefore$ 3 = 0.5 $\times$ v i.e. initial velocity = 6ms^{-1}	A1
	$a = v\dfrac{dv}{dx} = -4(1+x)$	B1 M1
	$\therefore \dfrac{v^2}{2} = \dfrac{-4(1+x)^2}{2} + C$	A1
	$\Rightarrow v^2 = -4(1+x)^2 + K$	
	v = 6 when x = 0 $\Rightarrow$ K = 40	A1 f.t.
	$\therefore v^2 = 40 - 4(1+x)^2$	
	When v = 0 $(1+x)^2 = 10 \Rightarrow x = 2.16$ m (3 S.F.)	M1 A1 ca.o
		[8]

MOTION UNDER VARIABLE ACCELERATION

9.	a) $a = -e^{-x}$	B1
	$v\dfrac{dv}{dx} = -e^{-x}$	M1
	$\int v\,dv = \int -e^{-x}\,dx \Rightarrow \dfrac{v^2}{2} = e^{-x} + C$	A1 A1
	Initially, v = 1, x = ln2	
	$\frac{1}{2} = e^{-\ln 2} + C \Rightarrow C = 0$	B1
	$v^2 = 2e^{-x}$	
	$v = \sqrt{2}\sqrt{e^{-x}} = \sqrt{2}e^{-\frac{x}{2}}$	A1 c.a.o.
		[6]
	b) $\dfrac{dx}{dt} = \sqrt{2}e^{-\frac{x}{2}}$	
	$\int e^{\frac{x}{2}}dx = \int \sqrt{2}dt$	M1
	$2e^{\frac{x}{2}} = \sqrt{2}t + C$	A1
	t = 0, x = ln2 $\Rightarrow 2\sqrt{2} = C$	M1 A1
	$2e^{\frac{x}{2}} = \sqrt{2}t + 2\sqrt{2}$	
	$e^{\frac{x}{2}} = \dfrac{\sqrt{2}t + 2\sqrt{2}}{2} = \dfrac{t+2}{\sqrt{2}}$	
	$\dfrac{x}{2} = \ln\left(\dfrac{t+2}{\sqrt{2}}\right)$	M1 A1
	$x = 2\ln\left(\dfrac{t+2}{\sqrt{2}}\right)$	
	$= \ln\left(\dfrac{t+2}{\sqrt{2}}\right)^2 = \ln\left(\dfrac{(t+2)^2}{2}\right)$	M1 A1
		[8]
	c) $v = \sqrt{2}e^{-\frac{x}{2}}$	
	$= \sqrt{2}e^{-\frac{1}{2}\ln\left(\frac{(t+2)^2}{2}\right)} = \sqrt{2}e^{\ln\left(\sqrt{\frac{2}{(t+2)^2}}\right)}$	M1 A1
	$= \dfrac{2}{t+2}$	A1 (or by diffn)
		[3]
	d) 0	B1
		[1]

MOTION UNDER VARIABLE ACCELERATION

10.a) $ma = -\frac{kv}{t}$

$\Rightarrow \frac{dv}{dt} = -\frac{k}{m}\frac{v}{t}$ B1 B1

[2]

b) $\int \frac{dv}{v} = -\frac{k}{m}\int \frac{dt}{t}$ M1

$\ln v = -\frac{k}{m}\ln t + C$ A1 A1

Taking exponentials : $v = e^{-\frac{k}{m}\ln t + C}$ M1

$v = t^{-\frac{k}{m}}e^{C}$ A1

so $L = \frac{k}{m}$, $A = e^{C}$

[6]

c) $\frac{dx}{dt} = At^{-L}$ M1

$\int dx = \int At^{-L}dt$

$L \neq -1 \Rightarrow \quad x = \frac{At^{-L+1}}{-L+1} + C$ A1

$t = 0, x = 0: \quad 0 = \frac{A \times 0^{-L+1}}{-L+1} + C$

$L < -1 \Rightarrow -L + 1 > 2$, so $0^{-L+1} = 0$ B1

so $C = 0$

$x = \frac{At^{1-L}}{1-L}$ A1

[4]

d) $t = 3, x = 18 \Rightarrow \quad 18 = \frac{A \times 3^{1-L}}{1-L}$ ①

$t = 3, v = 18 \Rightarrow \quad 18 = A3^{-L}$ ② M1 A1

Hence $A = 18(3^{L})$ M1 (attempt to solve)

Substituting into ①:

$18 = \frac{18(3^{L})(3^{1-L})}{1-L}$ M1 (powers)

$18 = \frac{18 \times 3}{1-L}$

$1 - L = 3 \Rightarrow \quad L = -2$ A1

$18 = A \times 3^{2} \Rightarrow A = 2$ M1 A1

[7]

MOTION UNDER VARIABLE ACCELERATION

QUESTION 10 CONTINUED

v) $x = \dfrac{2t^3}{3}$

$\left.\begin{array}{l} t = 2 \Rightarrow x = \dfrac{16}{3} \\ t = 4 \Rightarrow x = \dfrac{128}{3} \end{array}\right\}$ M1 A1

Distance $= 37\frac{1}{3}$ A1

[3]

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11. a) $|\mathbf{r}|^2 = 9\cos^2 2t + 9\sin^2 2t$ M1 A1

$|\mathbf{r}|^2 = 9 \qquad |\mathbf{r}| = 3$ A1

implies particle always 3 m from 0 $\therefore$ locus is a circle radius 3m. A1

[4]

b) $\mathbf{v} = \dfrac{d\mathbf{r}}{dt} = -6\sin 2t\,\mathbf{i} + 6\cos 2t\,\mathbf{j}$ M1 A1

$|\mathbf{v}|^2 = 6^2\sin^2 2t + 6^2\cos^2 2t$ A1

$|\mathbf{v}| = 6$ i.e. speed is constant and 6 ms^{-1} A1

[4]

c) Initially $\mathbf{r} = 3\mathbf{i} + 0\mathbf{j}$ B1

Returns to A when $\cos 2t = 1$ and $\sin 2t = 0$ M1

i.e. $2t = 2\pi \qquad t = \pi$ A1

[3]

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12. a) $\mathbf{v} = \dfrac{d\mathbf{r}}{dt} = -6\sin 2t\,\mathbf{i} - 4\cos 2t\,\mathbf{j}$ M1 A1

$\mathbf{a} = \dfrac{d\mathbf{v}}{dt} = -12\cos 2t\,\mathbf{i} + 8\sin 2t\,\mathbf{j}$ M1 A1 f.t.

[4]

b) $-12\cos 2t\,\mathbf{i} + 8\sin 2t\,\mathbf{j} = -4(3\cos 2t\,\mathbf{i} - 2\sin 2t\,\mathbf{j})$

$\therefore \quad \mathbf{a} = -4\mathbf{r}$ i.e. $\mathbf{a} = k\mathbf{r} \quad k = -4$ M1 A1

[2]

c) $x = 3\cos 2t \qquad y = -2\sin 2t$ B1

$\cos 2t = \dfrac{x}{3} \qquad \sin 2t = -\dfrac{y}{2}$ M1

$\cos^2 A + \sin^2 A = 1 \quad \Rightarrow \quad \dfrac{x^2}{9} + \dfrac{y^2}{4} = 1$ A1

[3]

MOTION UNDER VARIABLE ACCELERATION

13.a) $v = 0 \;\Rightarrow\; \cos t = \sqrt{3}\sin t$	M1
$\frac{1}{\sqrt{3}} = \tan t \;\Rightarrow\; t = \frac{\pi}{6}$	M1 A1
$a = \frac{dv}{dt} = -\sin t - \sqrt{3}\cos t$	M1 A1
$\Rightarrow a = -2$	A1
$\therefore$ acceleration of magnitude 2 ms^{-2}, in direction of x decreasing.	B1
	[7]
b) $x = \int v \, dt$	M1
$\therefore \; x = \sin t + \sqrt{3}\cos t + C$	A1
When $t = 0$ $x = 0$ $\therefore$ $C = -\sqrt{3}$	A1 f.t.
$\therefore \; x = \sin t + \sqrt{3}\cos t - \sqrt{3}$	A1
	[4]

14.a) $\mathbf{v} = \frac{d\mathbf{r}}{dt} = 6e^{2t}\,\mathbf{i} + 4e^{-2t}\,\mathbf{j}$	M1 A1
	[2]
b) $\mathbf{a} = \frac{d\mathbf{v}}{dt} = 12e^{2t}\,\mathbf{i} - 8e^{-2t}\,\mathbf{j}$	M1 A1 f.t.
$\mathbf{F} = m\mathbf{a} \;\Rightarrow\; \mathbf{F} = 36e^{2t}\,\mathbf{i} - 24e^{-2t}\,\mathbf{j}$	B1
$\Rightarrow \mathbf{F} = 12\,\mathbf{r}$	A1 f.t.
$\therefore$ force always in direction of $\overrightarrow{OP}$.	B1
$t = 0 \;\Rightarrow\; \mathbf{F} = 36\mathbf{i} - 24\mathbf{j}$	M1
$\lvert \mathbf{F} \rvert = \sqrt{36^2 + 24^2} = 43.3$ (1.d.p)	M1 A1 ca.o.
	[8]
c) $x = 3e^{2t}$ $\quad y = -2e^{-2t}$	B1
$e^{2t} = \frac{x}{3}$ $\quad e^{2t} = -\frac{2}{y}$	M1 A1
$\therefore \; \frac{x}{3} = -\frac{2}{y}$ $\quad xy = -6$	M1 A1
	[5]

MOTION UNDER VARIABLE ACCELERATION

15. a) $t = 0 \Rightarrow \mathbf{r} = (2 + \sqrt{3})\mathbf{i}$	M1 A1
$\therefore \ \lvert \mathbf{r} \rvert = 2 + \sqrt{3} \Rightarrow$ distance $= (2 + \sqrt{3})$m	A1 f.t.
	[3]
b) i) $\mathbf{v} = \dfrac{d\mathbf{r}}{dt}$ and $\mathbf{a} = \dfrac{d\mathbf{v}}{dt}$	M1
$\mathbf{v} = -\sqrt{3}\sin t\,\mathbf{i} + \cos t\,\mathbf{j}$	A1 ca.o.
$\mathbf{a} = -\sqrt{3}\cos t\,\mathbf{i} - \sin t\,\mathbf{j}$	A1 f.t.
	[3]
ii) $\tan\theta = \dfrac{\cos t}{-\sqrt{3}\sin t} = -\dfrac{1}{\sqrt{3}}\cot t$	M1 A1
$\tan\phi = \dfrac{-\sin t}{-\sqrt{3}\cos t} = \dfrac{1}{\sqrt{3}}\tan t$	A1
	[3]
iii) $\alpha = \phi - \theta$	B1
$\Rightarrow \tan\alpha = \dfrac{\tan\phi - \tan\theta}{1 + \tan\phi\tan\theta}$	M1
So $\tan\alpha = \dfrac{\frac{1}{\sqrt{3}}(\tan t + \cot t)}{1 - \frac{1}{3}}$	A1 A1
$= \dfrac{\sqrt{3}}{2}(\tan t + \cot t)$	A1
	[5]
iv) $\tan\alpha = \dfrac{\sqrt{3}}{2}\left(\dfrac{2}{\sin 2t}\right) = \dfrac{\sqrt{3}}{\sin 2t}$	M1
Perpendicular $\Rightarrow \alpha = 90^{\circ}$	B1
$\tan\alpha$ infinite	B1
$\Rightarrow \sin 2t = 0$	M1 A1
True when $t = 0$, $t = \dfrac{\pi}{2}$	A1 A1
	[7]
c) $\lvert \mathbf{v} \rvert = 3\sin^2 t + \cos^2 t = 1 + 2\sin^2 t$	M1 A1
Max $\lvert \mathbf{v} \rvert^2 = 3$ min $\lvert \mathbf{v} \rvert^2 = 1$	M1
Max $\lvert \mathbf{v} \rvert = \sqrt{3}$ min $\lvert \mathbf{v} \rvert = 1$	A1
	[4]